Some General Results of Elementary Sampling Theory for Engineering Use

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EVERY day we base conclusions on the results of the process commonly known as "sampling." For example, if five times in a week a man has waited ten minutes or more for his trolley at a street corner, he may conclude that the transportation facilities are poor. Or again, if a housewife has bought ten loaves of bread at a certain store and has found five of them not as fresh as might be desired, she decides that in the future she will buy her bread elsewhere. Both of these conclusions are based on an intuitive application of sampling theory. Such examples could be multiplied indefinitely.

Similarly, in most engineering problems, observational data are involved in one way or another. In order to be able to assign the proper significance to these data, it is essential to have some idea as to their reliability, that is, to what extent they represent all the facts under consideration. First, the measurements themselves may be in error. In the second place, although the observations may have been made with perfect precision, they may be incomplete; they may constitute but a "sample" of a large group of possible observations. The problem considered in this paper is one of this second class, generally known as "sampling" problems.

Assume the existence of a total group or "universe" of N objects and that observations have been made on a certain number n of them with reference to a particular characteristic. This number n we will call the "sample." From this sample we wish to deduce some estimate concerning the probable condition of that universe with reference to the characteristic observed.

Now the characteristic observed may itself take on one of two forms. It may be either, (1) present or absent; (2) quantitative. For simplicity in discussion we may call the first, "Sampling of Attributes," and the second, "Sampling of Variables."

An example of each will be cited from the telephone field.

Example 1: Sampling of Attributes

Suppose that 4,000 relays of a particular type constitute a day's output. In order to determine roughly what proportion of these are non-operative at a current of 12 mils, a sample of 500 relays is tested and out of this sample 10 fail to operate at the required current. In

the sample, then, two per cent of the relays were defective. What, then, is the probability that the percentage of the 4,000 relays having this defect is between one and three per cent? Or what is the probability that the percentage of defectives in the universe of 4,000 does not exceed four per cent? Or again, if we wish to be practically certain that among the 4,000 relays not more than two per cent are defective in this respect, how many defectives would be allowable in a sample of 200? or a sample of 1,000? Any number of questions of this sort can be asked and may be answered on the basis of the proper assumptions by sampling theory.

Example 2: Sampling of Variables

An office serves 5,000 subscribers lines. Measurements of the insulation resistance are made on 200 of these, selected at random, and the resulting values tabulated. They vary all the way from 12,000 ohms to 200,000 ohms. What conclusions may be drawn as to the probability that more than a certain number, say 20 of the subscribers' loops out of the 5,000, have an insulation resistance of less than 18,000 ohms? What is the most probable distribution of the insulation resistances for the office as a whole? What is the probable error of the average of the observations as a measure of the average loop insulation resistance for the office?

As before, much information *regarding the universe* may be inferred from a properly chosen sample, always, however, with some degree of uncertainty. This uncertainty, so far as the sampling process is concerned, naturally decreases as the size of the sample increases, and, of course, disappears except for inaccuracies of measurement, when the sample becomes coextensive with the universe.

The respective treatments of these two types of problems differ considerably in detail. The basic principles are, however, essentially the same, and involve in each case the notions of "a posteriori" probability, as discussed in most of the standard textbooks on the theory of probability.

In both problems there are certain observations. By means of these we desire to obtain as precise information as possible concerning some one or more characteristics of the universe from which these observations or samples were drawn. The true nature of the universe is to some degree, at least, unknown. Certain hypotheses concerning it may, however, in the light of the sample be more probable than others. What we wish to estimate is the probability that either a particular hypothesis or a group of mutually exclusive hypotheses includes the true one.

This article will be devoted to the type of problem termed "Sampling of Attributes."[1] In it are included results from an extensive series of computations in the form of charts which may be of value in the solution of practical engineering problems. The nomenclature is general, so as to be applicable to a wide variety of practical problems. For convenience in discussion we shall divide the units of any sample into the two mutually exclusive classes, "defective" and "satisfactory." The following notation will be used:

N = total number of items in universe,

n = total number of items in sample,

X = number of defective items in universe (unknown),

c = number of defective items in sample (observed),

$w(X)$ = *a priori* probability that the universe will contain exactly X defectives,

$W(X_1, X_2)$ = *a posteriori* probability that the universe contains a number of defectives X such that $X_1 \leqq X \leqq X_2$.

It is of extreme importance that, at the outset, the significance of the symbol $w(X)$ in sampling problems be clearly defined. It is a measure of the probability, *before the sample* is taken, that the lot or universe in question contains X defective items and $N - X$ satisfactory items. It may be based on previous samples, or the reputation of the manufacturer producing those items, or on any one or more of a number of other pertinent data. For example, even before a sampling inspection, we should unhesitatingly say that in a lot of 1,000 relays sent out by a reputable manufacturer it is very much more likely, *a priori*, that the lot will contain less than 100 relays with a short-circuited winding than that the lot will contain more than 800 relays defective in the same respect. We should probably find ourselves in a quandary, however, if we attempted to state without a sample inspection, the relative likelihoods of 3, 4, 5, 6, $\cdots$, etc., defectives existing in the lot. $w(X)$ is a function whose numerical value is assumed to state this *a priori* probability. The extent to which we are able to make use of this function, then, depends on how precisely we are able to assign numerical values to it before we study our sample.

[1] This general type of problem has been under study within the Bell System for some time. In an article "Deviation of Random Samples from Average Conditions and Significance to Traffic Men" by E. C. Molina and R. P. Crowell which appeared in the *Bell System Technical Journal* for January, 1924, a special case of sampling theory was developed and various possible applications were suggested. In August, 1924, Molina delivered a paper entitled "A Formula for the Solution of Some Problems in Sampling" before the statistical section of the International Mathematical Congress in Toronto, Canada. This paper dealt with a somewhat more general case of the sampling problem than was discussed in the article just mentioned.

It may be helpful at this point to state and solve a simple problem, which will serve to bring out the fundamental principles involved. An urn is known to contain 10 balls, some of which are white and the others black. Five balls have been drawn and *not* replaced. Of these five, one is white and four are black. What is now the probability that the urn originally contained just one white ball and nine black? Two white and eight black?

Before we proceed to obtain a solution for this problem we have to make some assumption, based on knowledge available before the drawings were made, concerning the probability that the urn contains black and white balls in any given proportion.

Consider two such assumptions—

(*a*) All proportions are *a priori* equally likely, i.e., before the drawings it is as likely that three whites and seven blacks were put in the urn as six whites and four blacks, etc.

(*b*) The urn was filled with ten balls drawn at random from a bag containing a very large number of balls of which a quarter are white and the remainder are black.

There are, before the drawings, 11 possible hypotheses concerning the contents of the urn. They range from 0 whites and 10 blacks to 10 whites and 0 blacks, as listed in the two left-hand columns of Table I and shown in Fig. 1. The probability in favor of each of

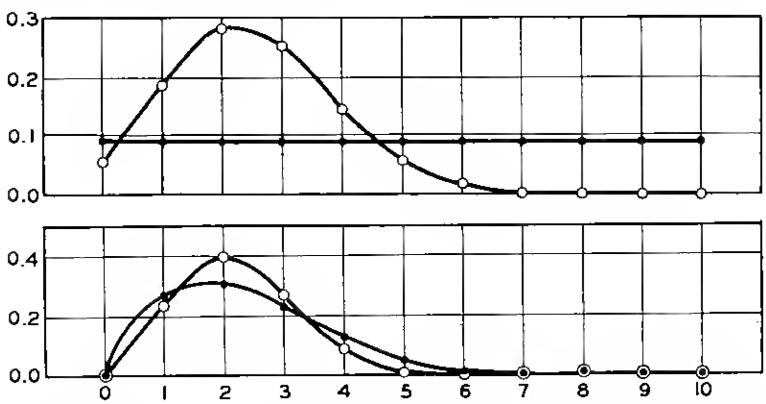


Fig. 1. The upper curve shows two different assumptions concerning the *a priori* probabilities, while the lower pair shows the *a posteriori* probabilities. In both cases the dots refer to the hypothesis of uniform *a priori* probability while the circles refer to the assumption that the urn itself is a random sample from a large stock of which one fourth of the balls are white.

these hypotheses is the "*a priori* existence probability" in favor of the hypotheses, and is represented by the symbol $w(X)$, X referring to the number of white balls assumed to be in the urn.

Under assumption "*a*" (Case 1) each hypothesis has a probability of 1/11 or .090909. Under assumption "*b*" (Case 2) the probability that the urn contains X whites and $10 - X$ blacks is the binomial term $\binom{10}{X}\left(\frac{1}{4}\right)^X\left(\frac{3}{4}\right)^{10-X}$.

TABLE I

Contents		Existence Prob. $w(X)$		Prod. Prob.	*A Posteriori* Prob. P_X	
"X" Wh.	Bl.	Case 1	Case 2	p_X	Case 1	Case 2
0	10	.090909	.056314	.000000	.000000	.000000
1	9	.090909	.187712	.500000	.272727	.237305
2	8	.090909	.281568	.555556	.303030	.395509
3	7	.090909	.250282	.416667	.227272	.263671
4	6	.090909	.145998	.238095	.129870	.087889
5	5	.090909	.058399	.099206	.054112	.014650
6	4	.090909	.016222	.023810	.012987	.000876
7	3	.090909	.003090	.000000	.000000	.000000
8	2	.090909	.000386	.000000	.000000	.000000
9	1	.090909	.000029	.000000	.000000	.000000
10	0	.090909	.000001	.000000	.000000	.000000

In the column headed p_X we give the productive probabilities for both cases. These are the probabilities that five drawings from an urn whose contents were as given by the corresponding hypothesis would yield the observed one white ball and four black. These are zero in the case of $X = 0, 7, 8, 9$ and 10 since urns so constituted could not have given the observed drawings.

For the other cases, the productive probability is the ratio

$$p_X = \frac{\binom{X}{1}\binom{10-X}{4}}{\binom{10}{5}}.$$

In this expression the denominator is the total number of combinations of 10 balls taken five at a time, and the numerator is the number of ways of selecting one out of X white balls and four out of the remaining $10 - X$ black balls. These figures are tabulated in Table I under the heading p_X.

We now have all of the component parts of our problem under the two different assumptions "*a*" and "*b*." It only remains to apply "Bayes' Rule."[2] Now the generalized Bayes Rule tells us that the *a posteriori* probability, P_X, in favor of an hypothesis *after* the drawings

[2] This rule was first enunciated by an English cleric, Bayes by name, in a memoir in *Philosophical Transactions* for 1763. It was generalized by Laplace in 1812 to cover cases not equally likely.

have been made and taking account of the *a priori* information is given by the ratio

$$P_X = \frac{w(X)p_X}{\sum w(X)p_X},$$

the summation in the denominator being extended over all possible cases.

The numerical values of this ratio are shown in the last two columns of Table I, corresponding to the two assumptions in our problem and also by the circles in Fig. 1. That we should have a different set of results corresponding to the different assumptions is to be expected. It is interesting, however, that the difference in this case is by no means great as Fig. 1 brings out.

If after each drawing we had replaced the ball drawn, we would have used for the productive probability p_X the binomial term

$$p_X = \binom{5}{1}\left(\frac{X}{10}\right)^1\left(\frac{10-X}{10}\right)^4$$

since the successive drawings would not have changed the relative constitution of the urn. The same would also be true if the urn contained an indefinitely large number of balls with the same relative proportions of black and white.

Now if we agree that a white ball corresponds to a defective item and a black ball to an acceptable item, we are immediately able, by the use of these fundamental principles of *a posteriori* probability, to write the general basic formal relation

$$W(X_1, X_2) = \frac{\sum_{X=X}^{X=X_2} w(X)\binom{X}{c}\binom{N-X}{n-c}}{\sum_{X=c}^{X=N-n+c} w(X)\binom{X}{c}\binom{N-X}{n-c}}. \qquad (1)[3]$$

As we have just indicated, the troublesome element in this formula is the function $w(X)$ to which, in many practical problems, it is difficult to assign any particular numerical values. In order to proceed further, therefore, without detailed consideration of various specific engineering problems we are forced to make some rather general assumptions concerning the nature of the function $w(X)$.

Case I

One of the most natural assumptions to make when no knowledge exists to the contrary is that $w(X)$ is a constant within that range

[3] It should be noted that in his original treatment of this formula Molina used S instead of Σ as the symbol for summation on account of the fact that finite integration entered into his analysis. Since in this presentation we are dealing only with summation, we shall use the commoner form Σ to denote summation.

of values of X which essentially affects the value of the denominator of (1). This assumption may seem at first glance rather arbitrary and wide of the mark, especially since the range of values which essentially affects the value of the denominator in (1) depends on the value of c obtained from the sample. However, if the sample is reasonably large, consisting of 100 items or more, and the proportion of defectives observed is small, say 10 per cent or under, the probability that universes having a proportion of defectives widely different from the one observed would yield such results is so small that a wide range of assumptions concerning the *a priori* probability of such universes existing makes very little change in the final result.

Applying, then, this assumption analytically to the basic formula (1) we obtain the simpler formulæ

$$W(X_1, X_2) = \frac{\sum_{X=X_1}^{X=X_2} \binom{X}{c}\binom{N-X}{n-c}}{\sum_{X=c}^{X=N-n+c} \binom{X}{c}\binom{N-X}{n-c}} = \frac{\sum_{X=X_1}^{X=X_2} \binom{X}{c}\binom{N-X}{n-c}}{\binom{N+1}{n+1}} \tag{2}$$

and by means of a transformation outlined in the Appendix we obtain, from (2),

$$W(X_1, X_2) = \frac{\sum_{t=0}^{t=c}\left[\binom{X_1}{t}\binom{N+1-X_1}{n+1-t} - \binom{X_2+1}{t}\binom{N-X_2}{n+1-t}\right]}{\binom{N+1}{n+1}}. \tag{2a}$$

Formula (2a) is the one embodied in the paper referred to in footnote 1. While apparently less simple than (2), it is actually easier to compute when c is less than the range $X_2 - X_1$.

When in (2a) we set $X_1 = c$ and $X_2 = X$ the resulting formula

$$W(c, X, n, N) = 1 - \frac{\sum_{t=0}^{t=c} \binom{X+1}{t}\binom{N-X}{n+1-t}}{\binom{N+1}{n+1}}, \tag{3}$$

which is at the basis of our computational work, shows explicitly certain properties which are not apparent in (2). Various analytical transformations and approximations based on this formula lead to several interesting extensions which are discussed in the Appendix. We shall leave these phases of the problem for the present, however, and discuss the results of the calculations which have been made as presented on the attached charts.

Charts A

Charts A have been prepared by means of exact formula (3) to show, for universes $N = 300$, 500, 700 and 900 from which samples, n, of various indicated sizes are assumed to have been drawn, the probability or "weight" $W(c, X)$ as ordinate versus X as absicissa for various values of c as indicated by the solid curves so designated. The dotted curves crossing these solid curves show the weight indicated by various values of the difference "d" between the percentage observed defective and the percentage assumed defectives in the universe.

As examples illustrating the interpretation of Charts A consider the following:

Example 1: From a universe of $N = 700$ items a random sample $n = 300$ items has shown $c = 3$ or 1 per cent defectives. What is the probability or weight to be associated with the hypothesis that the universe contains not more than $X = 14$ or two per cent defectives? From the A Chart corresponding to $N = 700$ and $n = 300$ we find the $c = 3$ curve (shown heavy because it is an even per cent of the sample $n = 300$). On this curve corresponding to an abscissa of $X = 14$ we read our desired result as the ordinate $W = .94$. We note that this is also a point on the $d = 1$ per cent dotted curve since $100(X/N - c/n)$ per cent $= 1$ per cent.

Example 2: We are going to make a sample of $n = 199$ items out of a universe of $N = 500$ items and wish the weight or probability to be .9 or better that the universe does not contain more than five per cent defective items. What is the maximum number of defective items that we may tolerate in our sample? Now five per cent of $N = 500$ is $X = 25$. Corresponding to an abscissa $X = 25$ and an ordinate $W = .9$ we locate a point which lies between the $c = 6$ and $c = 7$ curves. We could, therefore, accept the lot provided the sample showed six or less defectives, or three per cent or less defectives.

These Charts A are fundamental in nature, and involve the five variables, N, n, X, c and W. The formula by means of which they were computed is exact on the basis of the assumptions. Such errors or irregularities as may appear to exist in them are of negligible practical importance in view of the nature of the assumptions made, and are mainly due to the difficulties in drafting such a family of curves.

Naturally a function involving several variables may be represented graphically in many different ways, some of which may be more convenient than others to use in connection with various practical problems. One of the restrictions often encountered in practical

problems is that the weight W shall not be less than some specified figure which may be considered to give us the desired degree of confidence in the efficacy of our sampling procedure in weeding out defective lots. Charts B and C are drawn up on the basis of three such specified figures which are of practical interest, $W = .75$, $W = .9$, and $W = .99$. Such restrictions enable us to show, without the large amount of labor which would be required without them, the results of calculations for a wider range of the other variables.

Chart B

Chart B shows roughly for the proportion of observed defectives $c/n = .01$, $.04$, and $.07$, the proportion of defectives in the universe which we may expect not to exceed with weights $W = .75$, $.9$ and $.99$ for various values of the sample n as abscissa and for $N = 300$, 500, 700, 900 and also the limit approached as N becomes infinite. This form of presentation serves to relate the present material to the earlier charts which accompanied the earlier article already mentioned as having appeared in the *Bell System Technical Journal* for January, 1924, and shows how with a given size of sample n and a given proportion of defectives observed, the larger the value of the universe N, the larger the variation which may be expected with any given degree of probability. As would be expected, we also see that when the size of the sample approaches the size of the universe, the range of uncertainty approaches 0 and our sample inspection becomes a complete inspection.

It will be noted that, up to the present point, we have not considered cases for $N > 1{,}000$. The exact formulæ become rather troublesome to compute for these larger values of N. Fortunately, however, various approximate methods outlined in the Appendix become sufficiently accurate to be of service in these cases.

Charts C

We have, therefore, by their aid when $N > 1{,}000$, prepared the Charts C which we believe will cover a rather wide range of the variables with sufficient precision to be of considerable practical value. The points shown by dots are believed to be accurate to the degree to which they are readable on the chart. For intermediate values and for other values of the trouble limit the discrepancies are indicated on the charts. One of these charts corresponds to each of the three following weights, $W = .75$, $W = .9$, and $W = .99$. As abscissa we show the per cent sample, $100\ n/N$. The ordinate scale is proportional to the number of items n in the sample. The same

proportionality factor K enters also in the ratio X/N which we designate as the trouble limit. We shall later discuss the purpose of this factor K in more detail. The understanding of the charts will be simplified, however, if we consider the case for $K = 1$ in which the charts become direct reading for the case of a trouble limit $X/N = .01$.

The values of c, the number of defective items observed in the sample, are shown as a family of curves marked $c = 0$, $c = 1$, $c = 2$, etc., sloping downward from left to right. Any point on the $c = 5$ curve, for example, on the Chart C for weight $W = .9$ shows the corresponding values of n as ordinate and n/N as abscissa which are necessary in order that this number of defectives may be accepted with a degree of assurance[4] indicated by $W = .9$ that the true proportion of defectives in the universe N is not greater than .01.

It will be readily noted that for every value of the universe N, there may be drawn a diagonal straight line through the origin whose ordinate for an abscissa of 100 per cent sample is equal to $n = N$. Certain representative N lines are drawn in on the charts in this manner, and as many more could be inserted as desirable. Thus, for a constant value of W and a constant value of X/N we have provided on Charts C a ready means of determining the relationships which must exist between the remaining variables N, n, and c.

As an example of the use of these charts for the case where $K = 1$, i.e., for $X/N = .01$, consider the following:

Example 3: In a sample of $n = 900$ out of a universe $N = 3{,}000$, what is the maximum number of defectives c that we may accept with an assurance of $W = .9$ or better that the true proportion of defectives in the universe is not greater than .01?

Referring to the Charts C for $W = .9$ and considering $K = 1$, we locate the point corresponding to an abscissa of 100 n/N per cent $= 90{,}000/3{,}000 = 30$ per cent, and an ordinate $n = 900$. We find that this lies on the diagonal straight line marked $N = 3{,}000\ K$ as it should and that it also lies between the $c = 5$ and $c = 6$ curves. From this we may infer that we may accept five defectives but not six in the above case.

We shall now proceed to explain the significance of the factor K and the cross-hatched areas beneath the $c = 0, 5, 10, 15$, etc., curves. The purpose of these features is to extend the application of Charts C to values of X/N other than .01. It may be noted from the mathematical analysis or from actual plotting of charts similar to Charts C, but for different values of X/N, that the general shape and spacing of the curves remains practically unchanged for any given value of W.

[4] This statement is not strictly true when we are dealing with non-integral values of X. In such cases the weights W shown on the Charts C are slightly too high.

In other words, the value of $W(c, X)$ depends mainly on the ratio n/N, and the values of X and c, and only in a secondary way on the absolute values of n and N. This being the case, if we make a given per cent sample of two different universes N and KN, the number of defectives c which we may allow in our sample out of the first universe N in order that our weight W may have a given value, .9 say, for the true proportion of defectives in this universe to be not greater than .01 is practically the same as the value of c that we may allow in the sample out of the second universe KN for the same weight W and a proportion of defectives $.01/K$. For values of $K > 1$ there is no appreciable change introduced in the location of the c curves on Charts C. For values of $K < 1$, some error is made. The magnitude of this error is indicated by the cross-hatched bands on the $c = 0$, 5, 10, 15, etc., curves. The lower boundaries of these bands were calculated to show the magnitude of the error introduced for the corresponding values of c when $K = .1$. The upper boundaries of these areas correspond to values of $K \geqq 1$. For other values of c only the upper boundaries of the corresponding bands are shown, the lower boundaries being easily deducible by visual interpolation to a sufficient degree of approximation for most practical purposes.

As examples which may serve to illustrate this sort of application of Charts C consider the following:

Example 4: A sample of $n = 5{,}000$ items has been drawn out of a universe of $N = 20{,}000$ items and $c = 15$ defectives were observed. May we assume with a weight $W = .9$ or more that the true proportion of defectives or trouble limit X/N is .005?

Here $.01/K$ is to equal .005 for our charts to apply. Therefore, $K = 2$. Our sample $n = 500 = 2{,}500K$ and our per cent sample is $100\ n/N = 25$ per cent. Corresponding then to an abscissa of 25 per cent and an ordinate of $2{,}500K$ on the $W = .9$ chart we locate a point between the $c = 19$ and $c = 20$ curves. We could have allowed, therefore, $c = 19$ defectives at the desired weight and trouble limit. Since we observed a smaller number of defectives than was allowed, our weight W is therefore greater than .9. As a matter of fact it is practically only slightly less than .99 as appears from the $W = .99$ chart when utilized in a corresponding manner.

Example 5: As our next example we shall attempt to determine what is the trouble limit which corresponds with $W = .9$ to the results of the sample of Example 4. On the $W = .9$ chart corresponding to an abscissa of 25 per cent we read from the $c = 15$ curve an ordinate of $2{,}015K$. But this must be our sample $n = 5{,}000$. We, therefore, determine K from the equation $2{,}015K = 5{,}000$ which gives $K = 2.48$. Hence, our corresponding trouble limit is

$$\frac{.01}{K} = \frac{.01}{2.48} = .0040.$$

So far our values of K have been greater than unity, so we have not had to consider our cross-hatched bands at all. In our next example we shall remedy this defect.

Example 6: What number of defective items c may be allowed in a sample of 200 items out of a universe of 500 items so that $W \geq .9$ corresponding to a trouble limit $X/N = .08$. Here $.01/K = .08$ $\therefore K = .125$. Our ordinate, therefore, is $200 = 1{,}600K$ and our abscissa is $200/500 \times 100 = 40$ per cent sample. The point corresponding to this on the $W = .9$ chart lies just below the $c = 12$ curve indicating at first glance that we could not accept 12 defectives in such a case. However, we note that $K = .125$ should be near the lower boundary of our cross-hatched band for $c = 12$ if such a band had been drawn in. From an inspection of the widths of the bands for $c = 10$ and $c = 15$ we correctly infer that our point determined by the 40 per cent sample and $1{,}600K$ would lie well within this band, and that after all we could accept 12 defectives in the example in question.

This example has been included merely to illustrate the interpretation of the bands shown on Charts *C*. It may be anticipated that in many if not most of the practical engineering problems only the upper boundaries of the bands need be used to obtain a degree of accuracy commensurate with the precision of the results desired and the applicability of the basic assumptions concerning randomness and the form of the *a priori* existence probability $w(X)$.

If it should be desired to extend the range of these charts to cover values of W other than those shown, this may be done by means of the methods outlined in the mathematical analysis, the particular method to be used depending on the degree of precision required.

The preceding pages have contained an outline of some of the theory and results based on the assumption that, within a range at least, all possible values of X, the unknown number of defectives, were *a priori*, that is, before the sample in question was made, equally likely. This assumption we mentioned as appropriate to consider in case we have no information to the contrary. The results may be also applicable to certain cases where we do have some information of a general sort, but which it is difficult to express analytically. However, it is by no means the only reasonable assumption to make concerning the form of $w(X)$ as it enters into the basic formula (1).

Case II

Another assumption is suggested by the following considerations which enter into many of the standard works on probability theory. Assume that the lot or universe in question was itself drawn at random from an extremely large stock or major universe in which the proportion of defective items was p. Under these conditions the *a priori* probability $w(X)$ that our universe of N items would contain exactly X defective items would be given by the expression

$$w(X) = \binom{N}{X} p^X (1 - p)^{N-X}.$$

Using this expression for $w(X)$ in the fundamental formula (1), we obtain, by a process given in detail under the heading Appendix, the formula

$$W(X_1, X_2) = \sum_{X=X_1}^{X=X_2} \binom{N-n}{X-c} p^{X-c}(1-p)^{N-n-X+c},$$

which for $X_1 = c$ and $X_2 = X$ reduces to

$$W(c, X) = \sum_{t=0}^{X-c} \binom{N-n}{t} p^t (1-p)^{N-n-t},$$

which is precisely the expression for the *a priori* probability that the remaining $N - n$ items which we did not inspect contain not more than the $X - c$ defectives which together with the c we have observed would assure us of a satisfactory universe.

In this form $W(c, X)$ turns out to be a simple binomial which, when $N - n$ is large and p is small, may be reasonably approximated by the Poisson Exponential Binomial Limit for which extensive curves and tables already exist[5] and will, therefore, not be included in this article.

In order to make practical use of the results of this assumption, we must have some knowledge of the appropriate value of the factor p in any given case. This factor should measure the probability that an item, selected at random, will, on inspection, prove to be defective. If a large number of tests have been made in the past on similar items, prepared by essentially the same process, the ratio of the total defectives observed to total items inspected in such tests may be a reasonable figure to use for p. In the case of many manufactured articles such a ratio ought not to be very difficult to obtain. In certain cases it might be necessary to allow for such factors as

[5] See article, "Probability Curves Showing Poisson's Exponential Summation," by George A. Campbell, *Bell System Technical Journal*, January, 1923.

trend, improved process of manufacture, changes in personnel and so on. In the final sampling of such complicated equipment as is involved in a completely installed telephone central office it may be necessary to take account also of breakage and such troubles due to shipping and setting up of the equipment which may introduce marked deviations from average conditions.

Such general considerations as these should determine whether or not we can safely assume any given value for p, and if so, what value. It will be evident on a little consideration that the assumed value for p need not be extremely precise for many practical applications.

Concerning the restrictions on the function giving the *a priori* probabilities, $w(X)$, it may be well to point out that this function is only defined for the positive integral values of X such that $0 \leq X \leq N$. Moreover, since probabilities are essentially positive, it cannot be negative for any of these values of X. Also since the composition of the lot is certainly comprised in this range of values of X, one has

$$\sum_0^N w(X) = 1.$$

The questions raised concerning the form of $w(X)$ are of particular importance in connection with the economic phases of sampling, that is, the relative costs of having satisfactory lots rejected and unsatisfactory lots accepted by the sampling process. These costs are, of course, dependent on the frequency with which given proportions of defects occur in the lots in practice, and a detailed consideration of these would itself warrant a separate treatment.

It is felt that the general methods outlined in this treatment, while not sufficiently detailed for immediate practical application to many of the problems in sampling of attributes, will nevertheless serve as a satisfactory basis for further work of a more specific nature.

APPENDIX

Case I: Assuming $w(X)$ is a constant and noting that[6]

$$\sum_{X=c}^{X=N-n+c} \binom{X}{c}\binom{N-X}{n-c} = \binom{N+1}{n+1},$$

the fundamental formula

$$W(X_1, X_2) = \frac{\sum_{X=X_1}^{X=X_2} w(X)\binom{X}{c}\binom{N-X}{n-c}}{\sum_{X=c}^{X=N-n+c} w(X)\binom{X}{c}\binom{N-X}{n-c}} \qquad (1)$$

[6] Netto, "Lehrbuch der Kombinatorik," p. 15, Eq. 11.

gives us, as one computing formula,

$$W(X_1, X_2) = \frac{\sum_{X=X_1}^{X=X_2} \binom{X}{c}\binom{N-X}{n-c}}{\binom{N+1}{n+1}}, \tag{2}$$

which is fairly manageable so long as the range X_1 to X_2 is not too great and we have tables for the logarithms of the factorials involved. When $X_2 - X_1$ is large compared with c, one may use the equivalent formula

$$W(X_1, X_2) = \frac{\sum_{t=0}^{t=c} \binom{X_1}{t}\binom{N+1-X_1}{n+1-t} - \binom{X_2+1}{t}\binom{N-X_2}{n+1-t}}{\binom{N+1}{n+1}}. \tag{2a}$$

This transformation may, as Molina has shown, be effected as follows:[7]

$$\sum_{X=X_1}^{X=X_2}\binom{X}{c}\binom{N-X}{n-c} = \sum_{X=X_1}^{X=N-n+c}\binom{X}{c}\binom{N-X}{n-c} - \sum_{X=X_2+1}^{X=N-n+c}\binom{X}{c}\binom{N-X}{n-c}.$$

Now

$$\begin{aligned}\sum_{X=X_2+1}^{X=N-n+c}\binom{X}{c}\binom{N-X}{n-c} &= \sum_{X=X_2+1}^{X=N-n+c}\binom{N-X}{n-c}\left(\sum_{t=0}^{t=c}\binom{X_2+1}{t}\binom{X-X_2-1}{c-t}\right)\\ &= \sum_{t=0}^{t=c}\binom{X_2+1}{t}\left(\sum_{X=X_2+1}^{N-n+c}\binom{N-X}{n-c}\binom{X-X_2-1}{c-t}\right)\\ &= \sum_{t=0}^{t=c}\binom{X_2+1}{t}\binom{N-X_2}{n+1-t}.\end{aligned}$$

Likewise

$$\sum_{X=X_1}^{X=N-n+c}\binom{X}{c}\binom{N-X}{n-c} = \sum_{t=0}^{t=c}\binom{X_1}{t}\binom{N+1-X_1}{n+1-t}.$$

If in (2a) we let $X_1 = c$ and $X_2 = X$, we obtain

$$W(c, X) = 1 - \frac{\sum_{t=0}^{t=c}\binom{X+1}{t}\binom{\overline{N+1}-\overline{X+1}}{n+1-t}}{\binom{N+1}{n+1}}, \tag{3}$$

from which we may compute $W(X_1, X_2)$ from the equation

$$W(X_1, X_2) = W(c, X_2) - W(c, X_1 - 1).$$

[7] See also Netto, "Lehrbuch der Kombinatorik," p. 12, Eq. 6; p. 15, Eq. 11.

A direct interpretation of the expression for $W(c, X)$ gives us the following interesting

Theorem A: The *a posteriori* probability that a universe of N items contains not more than X defectives when c defectives have resulted from a random sample of n items is equal to the *a priori* probability of obtaining at least $c + 1$ defectives in a random sample of $n + 1$ items from a universe of $N + 1$ items of which exactly $X + 1$ are defective. This theorem assumes that *a priori* all values of X are equally likely.

Writing $s = X + 1 - t$, we obtain from (3)

$$1 - W(c, X) = 1 - \frac{\displaystyle\sum_{s=0}^{s=X-c} \binom{X+1}{s} \binom{\overline{N+1} - \overline{X+1}}{N - n - s}}{\displaystyle\binom{N+1}{N-n}}. \tag{4}$$

The right-hand side of this equation is exactly what would have been obtained directly from (3) if we had been dealing with a sample of $N - n - 1$ instead of n and had observed $X - c$ defectives instead of c, since the particular symbol chosen for the variable of summation is immaterial.

This fact, which follows immediately from physical consideration of the equivalent *a priori* problem of Theorem A, may be stated as

Theorem B: If we calculate the probability W that a universe of N items contains not more than X defectives when a sample of n has shown exactly c defectives, then $1 - W$ is the probability that a universe of N items contains not more than X defectives when a sample of $N - n - 1$ has shown exactly $X - c$ defectives.

In making extensive calculations, this relation will serve to cut down the amount of computation considerably, as each calculated value of W may be made to do double duty. For a single calculation either (3) or (4) may be used depending on which involves the shorter summation.

Another interesting relation also appears when we note that

$$\frac{\displaystyle\binom{X+1}{t}\binom{N-X}{n+1-t}}{\displaystyle\binom{N+1}{n+1}} = \frac{\displaystyle\binom{n+1}{t}\binom{N-n}{X+1-t}}{\displaystyle\binom{N+1}{X+1}},$$

which may be proved simply by cross multiplication of the combination factors, writing them in terms of factorials. From this we see that

$$W(c, X) = 1 - \frac{\displaystyle\sum_{t=0}^{t=c} \binom{n+1}{t} \binom{\overline{N+1} - \overline{n+1}}{X + 1 - t}}{\displaystyle\binom{N+1}{X+1}}. \tag{5}$$

If we compare the right-hand sides of (3) and (5), we see that n and X have simply been interchanged, which proves the rather interesting

Theorem C: The probability $W(c, X)$ that a universe of N items does not contain more than X defectives when a sample of n has shown exactly c defectives is equal to the probability $W(c, n)$ that a universe of N items does not contain more than n defectives when a sample of X has shown exactly c defectives.

Thus equations (3), (4) and (5) taken together show that we may make three different interpretations of a single calculation.

Up to this point, all of the analysis has been *exact* on the basis of the fundamental assumptions. We may now proceed with advantage to consider some approximate relationships which have been for some years of service in the calculation of practical curves and tables for cases where the values of N, n, or X were too great to be handled conveniently by means of the exact formulæ.

Now consider in formula (3) a single term, π_t say, where

$$\begin{aligned}\pi_t &= \binom{X+1}{t}\frac{\binom{N-X}{n+1-t}}{\binom{N+1}{n+1}} \\ &= \binom{X+1}{t}\frac{\left(\frac{(n+1)!}{(n+1-t)!}\right)\left(\frac{(N-n)!}{(N-n-X-1+t)!}\right)}{\left(\frac{(N+1)!}{(N-X)!}\right)} \\ &= \binom{X+1}{t}\left(\frac{n}{N}\right)^t\left(1-\frac{n}{N}\right)^{X+1-t}\cdot F(N, n, X, t), \end{aligned} \tag{6}$$

where the form of the function F is to be determined.

To facilitate the consideration of this function we may split it up into three similar parts as follows:

$$F(N, n, X, t) = \frac{\varphi(n+1, t-1)\cdot\chi(N-n, X-t)}{\varphi(N+1, X)},$$

where

$$\varphi(n+1, t-1) = \left(1+\frac{1}{n}\right)\left(1\right)\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{t-2}{n}\right),$$

$$\varphi(N+1, X) = \left(1+\frac{1}{N}\right)\left(1\right)\left(1-\frac{1}{N}\right)\cdots\left(1-\frac{X-1}{N}\right),$$

$$\chi(N-n, X-t) = \left(1\right)\left(1-\frac{1}{N-n}\right)\cdots\cdots\cdots\left(1-\frac{X-t}{N-n}\right).$$

Recalling that

$$\log (1 + X) = X - \frac{1}{2}X^2 + \frac{1}{3}X^3 - \frac{1}{4}X^4 + \cdots,$$

which converges for $X^2 < 1$,

we have

$$\log \varphi(n + 1, t - 1) = \log \left(1 + \frac{1}{n}\right) - \sum_{y=0}^{t-2} \sum_{r=1}^{\infty} \frac{1}{r}\left(\frac{y}{n}\right)^r,$$

$$\log \varphi(N + 1, X) = \log \left(1 + \frac{1}{N}\right) - \sum_{y=0}^{X-1} \sum_{r=1}^{\infty} \frac{1}{r}\left(\frac{y}{N}\right)^r,$$

$$\log \chi(N - n, X - t) = - \sum_{y=0}^{X-t} \sum_{r=1}^{\infty} \frac{1}{r}\left(\frac{y}{N - n}\right)^r,$$

whence

$$\log F(N, n, X, t) = \log \left(1 + \frac{1}{n}\right) - \log \left(1 + \frac{1}{N}\right)$$
$$+ \sum_{y=0}^{X-1} \sum_{r=1}^{\infty} \frac{1}{r}\left(\frac{y}{N}\right)^r - \sum_{y=0}^{t-2} \sum_{r=1}^{\infty} \frac{1}{r}\left(\frac{y}{n}\right)^r - \sum_{y=0}^{X-t} \sum_{r=1}^{\infty} \frac{1}{r}\left(\frac{y}{N - n}\right)^r$$

Neglecting terms of the second order in $1/n$, $1/N$ and $1/(N - n)$, we have as an approximation

$$\log F(N, n, X, t) \doteq -\frac{1}{2}\left(\frac{t^2 - 3t}{n} + \frac{X^2 - 2Xt + t^2 + X - t}{N - n} - \frac{X^2 - X - 2}{N}\right).$$

If now we select as our value of t, $t = (n/N)X$, we have $\log F(N, n, X, t) \doteq (X - 2)/2N$ which is > 0; if $X > 2$,

$$F(N, n, X, t) \doteq e^{(X-2)/2N},$$

which gives us as an approximate value for the maximum term π_t, where $t = (n/N)X$,

$$\pi_t' = \binom{X + 1}{t}\left(\frac{n}{N}\right)^t\left(1 - \frac{n}{N}\right)^{X+1-t} e^{(X-2)/2N}. \tag{7}$$

Having this term, it is a simple matter to calculate the other terms necessary for evaluating $W(c, X)$ by means of the exact equations

$$\pi_{t+1} = \pi_t\left(\frac{n - t + 1}{t + 1} \cdot \frac{X - t + 1}{N - n - X + t}\right), \tag{8}$$

$$\pi_{t-1} = \pi_t\left(\frac{t}{n - t + 2} \cdot \frac{N - n - X + t - 1}{X - t + 2}\right). \tag{9}$$

Due to the reciprocal relationship between n and X, we may obtain in a similar manner

$$\binom{n+1}{t}\frac{\binom{N-n}{x+1-t}}{\binom{N+1}{X+1}} = \binom{n+1}{t}\left(\frac{X}{N}\right)^t\left(1-\frac{X}{N}\right)^{n+1-t} e^{(n-2)/2N}, \quad (10)$$

when $t = (X/N)n$.

It is by means of these relationships that we have calculated the cases for $N > 1{,}000$ as shown on Charts C and feel that the precision obtained is rather better than would have resulted from using formula (6) for all values of t and assuming $F(N, n, X, t) = 1$. However, for suitable ranges of the variables involved, the formula resulting from this procedure

$$W(c, X) \doteq 1 - \sum_{t=0}^{t=c}\binom{X+1}{t}\left(\frac{n}{N}\right)^t\left(1-\frac{n}{N}\right)^{X+1-t} \quad (11)$$

would be a fairly good approximation. This is simply part of the well-known binomial expansion and is far simpler to compute than the more precise formulæ, although by no means easy at that.

We may draw several interesting practical conclusions, however, from formula (11). For instance, we may note that as n/N approaches 0 and X becomes infinite in such a way that the product $(X + 1)(n/N)$ remains constant and equal to the average a, we have the familiar Poisson Exponential Binomial Limit

$$W(c, X) = 1 - \sum_{t=0}^{c}\frac{a^t e^{-a}}{t!},$$

where $a = (X + 1)(n/N)$.

In addition we note from formula (11) that, for small values of X/N, the variable N enters into the formula only in the ratio n/N. From this we deduce the fact, borne out by independent calculations, that by means of the proper use of a proportionality factor K applied directly to n and N and inversely to X/N we may extend the Charts C to care for values of $X/N \leqq .1$ to a very good degree of accuracy and with considerable saving in space and computational labor.

By the reciprocal relationship between X and n as shown in exact formulæ (3) and (5), we obtain

$$W(c, X) \doteq 1 - \sum_{t=0}^{t=c}\binom{n+1}{t}\left(\frac{X}{N}\right)^t\left(1-\frac{X}{N}\right)^{n+1-t}, \quad (12)$$

which differs only in form from equation (3) of Molina's paper [8] on

[8] Footnote 1.

the infinite universe case. Formula (12) does not give the same results as (11) as it is most exact when n/N is small and becomes absolutely exact in the limiting case of an infinite universe where $n/N \doteq 0$. This formula also approaches the Poisson Limit, in this case as X/N approaches 0 and $n + 1$ becomes infinite in such a way that the product $(n + 1)(X/N)$ remains constant and equal to α, say.

The Poisson Limit, for the case of an infinite universe, was given by Molina in the Appendix to the article in the *Bell System Technical Journal* of January, 1924, already mentioned in this memorandum.

Another point of interest is brought out when we note that in the limiting form of (12) the Poisson gives us

$$W(c, X) \doteq \sum_{t=c+1}^{\infty} \frac{a^t e^{-a}}{t!}, \quad a = \frac{X \cdot n}{N},$$

and for another pair of values of W and X

$$W_1(c, X_1) = \sum_{t=c+1}^{\infty} \frac{a_1{}^t e^{-a_1}}{t!}, \quad a_1 = \frac{X_1 \cdot n}{N}.$$

Thus from properly chosen Poisson curves or tables we may obtain the ratio $X_1/X \doteq a_1/a$ which corresponds to the observed value of c and the desired values of W and W_1. This ratio in exact formulæ is a function of N, n, and X also, but for many problems involving small values of n/N and X/N the degree of approximation furnished by this limiting form is fairly satisfactory and still further reduces the amount of labor necessary in extending approximate results to practice.

The sort of procedure we have just been discussing may be facilitated by means of a chart on which we show as abscissæ values of c and as ordinates values of the ratio of X_1/X which corresponds to various values of W as shown by various curves and a specified value of W_1, say $W_1 = .9$. Such a chart would enable us to interpret roughly a given Chart C for $W = .9$ in terms of other values of W. For precise work this procedure is not to be recommended, and, therefore, no charts of the nature just described are included herein.

Approximations to the binomial other than the Poisson have been discussed in many of the texts. In particular, for values of p in the neighborhood of $\frac{1}{2}$, the well-known Laplace-Bernoulli integral

$$\frac{1}{\sqrt{\pi}} \int_a^b e^{-t^2} dt$$

will serve as an approximate value for W_1 where the limits a and b

are functions of N, n, X, and c. This approximation is not so suitable, however, for most telephone sampling problems in which the proportion of defectives may be assumed in general to be far smaller than $\frac{1}{2}$.

We shall now proceed to discuss a few points concerning the analysis of Case II in which instead of assuming $w(X)$ constant we assumed it to be of the form

$$w(X) = \binom{N}{X} p^X (1 - p)^{N-X}.$$

Combining this expression for $w(X)$ with the term $\binom{X}{c}\binom{N-X}{n-c}$ which appears in the basic formula (1), we have

$$\frac{X!}{c!(X-c)!} \cdot \frac{(N-X)!}{(n-c)!(N-X-n+c)!} \frac{N!}{X!(N-X)!} p^X (1-p)^{N-X}$$
$$= \binom{N}{n} \cdot \binom{n}{c} p^c (1-p)^{n-c} \cdot \left(\binom{N-n}{X-c} p^{X-c} (1-p)^{N-n-X+c} \right).$$

Since only the factors in brackets involve the variable of summation X, the remainder of this expression will cancel out in numerator and denominator, leaving us with

$$W(X_1, X_2) = \frac{\sum_{X=X_1}^{X=X_2} \binom{N-n}{X-c} p^{X-c}(1-p)^{N-n-X+c}}{\sum_{X=c}^{X=N-n+c} \binom{N-n}{X-c} p^{X-c}(1-p)^{N-n-X+c}}$$

as the resulting form for (1) with this assumption for $w(X)$.

It may be noted that the summation in the denominator above is a complete binomial $(p + q)^{N-n}$ and as such equals unity, so

$$W(X_1, X_2) = \sum_{X=X_1}^{X=X_2} \binom{N-n}{X-c} p^{X-c}(1-p)^{N-n-X+c},$$

where p is assumed to be the *a priori* probability of a defective item as determined from reliable information concerning conditions under which the items are prepared.

As before when $X_1 = c$ and $X_2 = X$ we have

$$W(c, X) = \sum_{t=0}^{t=X-c} \binom{N-n}{t} p^t (1-p)^{N-n-t}.$$

We may be willing in certain cases to admit the binomial form for

$w(X)$ without being able or willing to assign any single value to p. In such cases we may, however, proceed to make assumptions concerning the probability that p has a given value. Let

$$s(X, p) = f(p) \binom{N}{X} p^X (1 - p)^{N-X};$$

then

$$w(X) = \int_0^1 s(x, p) dp = \binom{N}{X} \int_0^1 f(p) p^X (1 - p)^{N-X} dp,$$

where

$$\int_0^1 f(p) dp = 1 \quad \text{and} \quad \sum_{X=0}^{N} w(X) = 1.$$

Suppose we assume $f(p)$ constant for all values between 0 and 1; we have

$$w(X) = \binom{N}{X} \int_0^1 p^X (1 - p)^{N-X} dp = \frac{1}{N + 1},$$

which we note to be a constant which assigns to all of the $N + 1$ possible *a priori* hypotheses concerning X an equal weight. This pair of assumptions in Case II amounts, therefore, to the same thing analytically as the assumption of Case I.

Any number of possible hypotheses concerning $f(p)$ might be made. Some of these would complicate the analysis considerably, others might be carried through fairly simply. One of these hypotheses might fit one class of physical problems, another some other class. To consider these all in detail in this paper would be outside of the scope of a general treatment. The methods outlined here would, however, hold for such extensions. Such difficulties as might be encountered would be of an analytical rather than a logical nature.

In closing, the author wishes to express his appreciation to his numerous friends and associates in the Bell System, whose suggestions and cooperation have been of material assistance in the preparation of this work, and particularly the work of Miss Nelliemae Z. Pearson of the Department of Development and Research, under whose direction most of the computations were carried out and who has checked through the various proofs.

Key to the Charts

The charts present various graphical representations of the function $W(c, X, n, N)$, equation 3. This function gives the probability, W, that the number of defectives in a lot of N is equal to *or less than* X, after a sample of n units has shown c defectives, *assuming that each of the possible values of X between o and N were equally likely a priori.*

Charts A: Separate pages refer to different values of n and N as labelled.

Ordinates, W; abscissas, X.

Solid curves, c; dotted curves $(c/n - X/N)$ expressed as per cent.

Charts B: Separate groups of curves refer to different values of W as labelled.

Ordinates, X/N; abscissas, n.

Separate sets of curves in each group refer to different values of c/n as labelled.

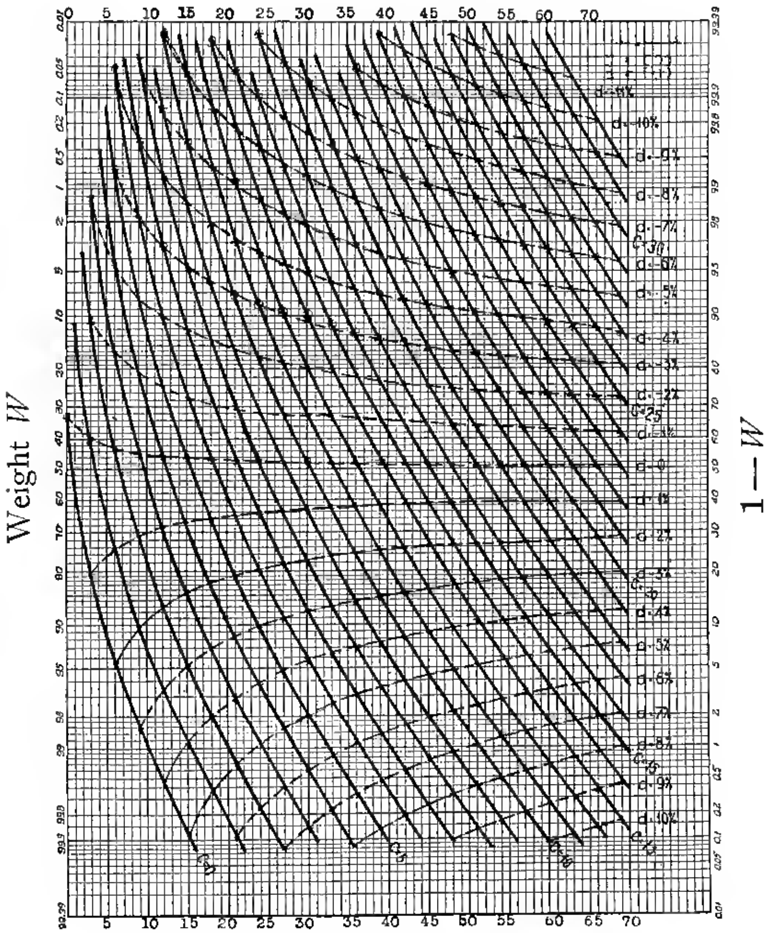
Individual curves are for different values of N.

Charts C: Separate pages refer to different values of W.

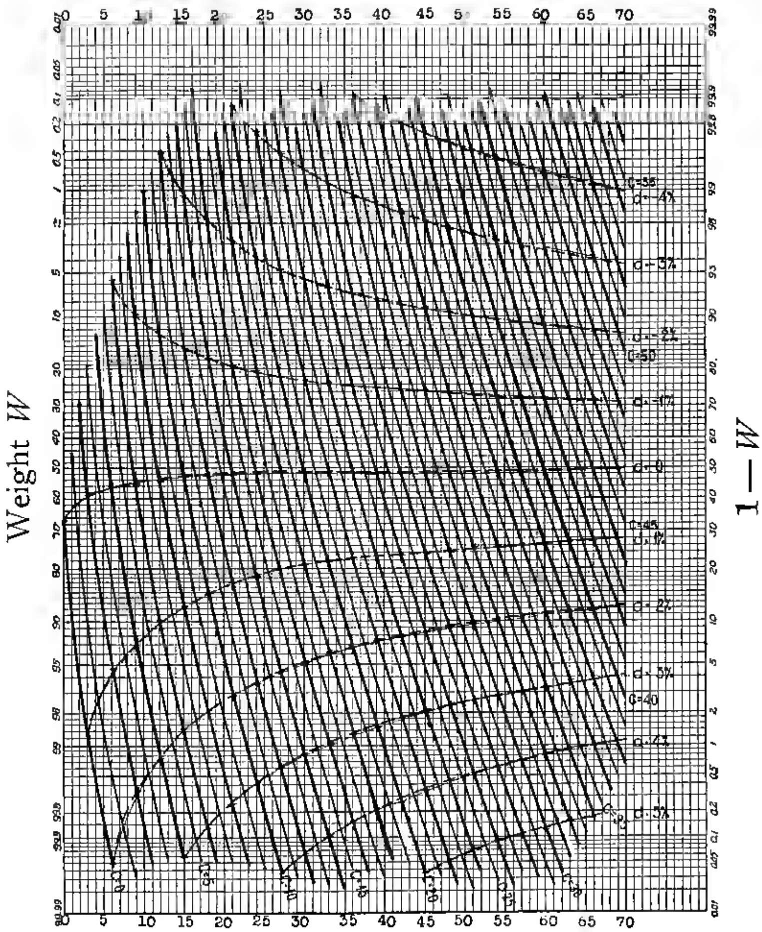
Ordinates, n; abscissas, n/N.

Separate curves for different values of c.

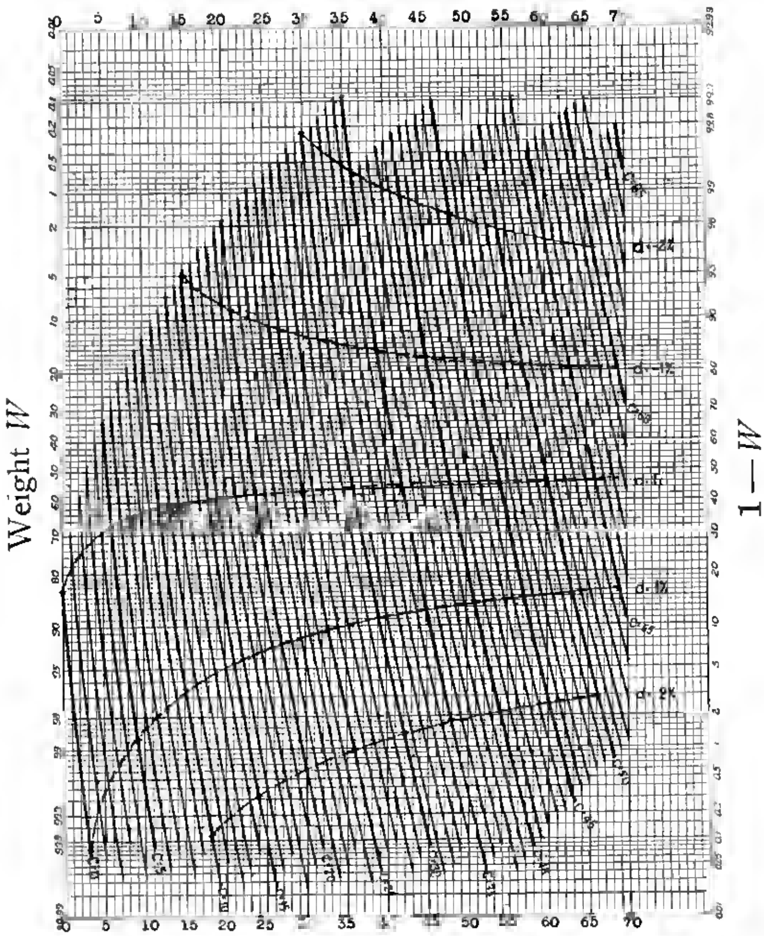
Cross-hatching indicates amount of dependence on X/N. For fuller explanation see pages 34–37 incl.



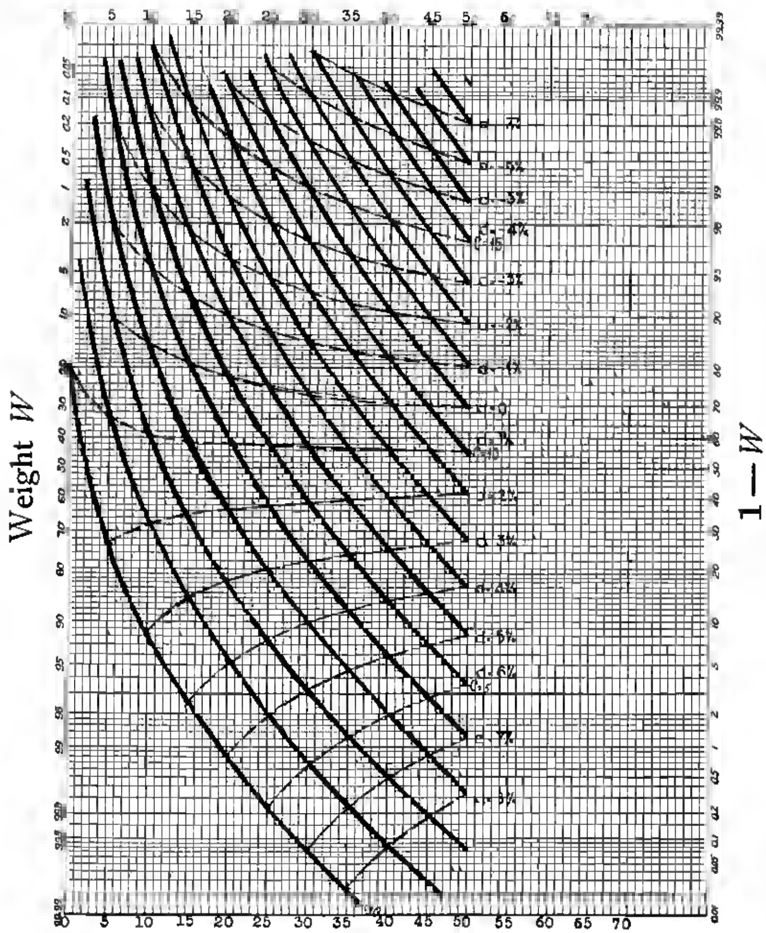
$X =$ Defectives in Universe
$N = 300,\ n = 100$
CHARTS A



X = Defectives in Universe
$N = 300$, $n = 199$
CHARTS A



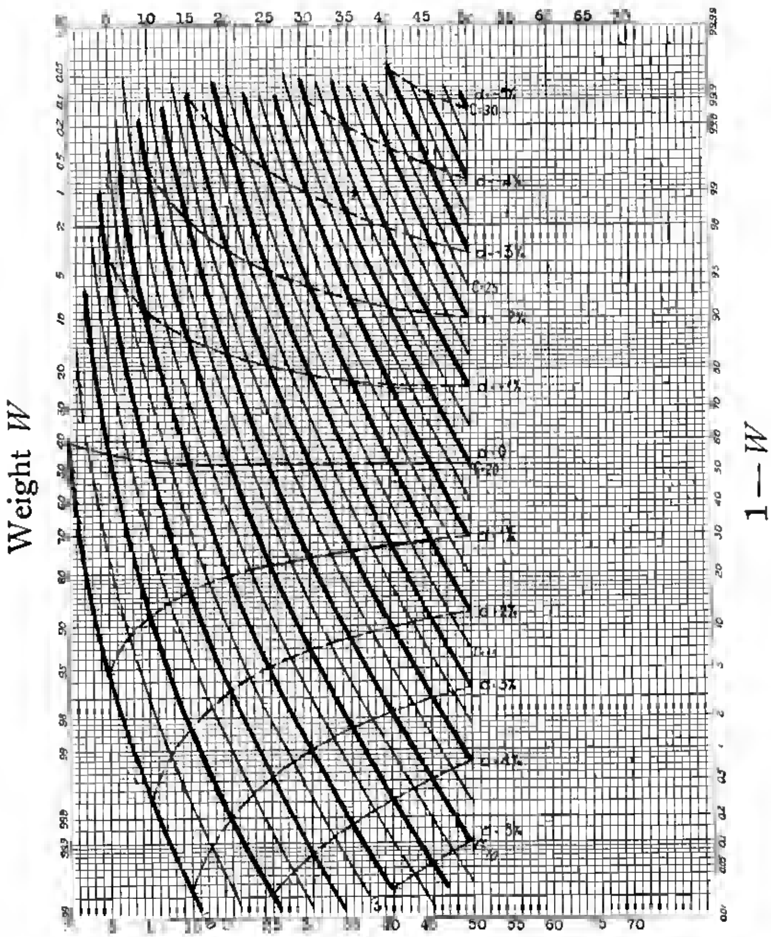
$X =$ Defectives in Universe
$N = 300,\ n = 249$
CHARTS A



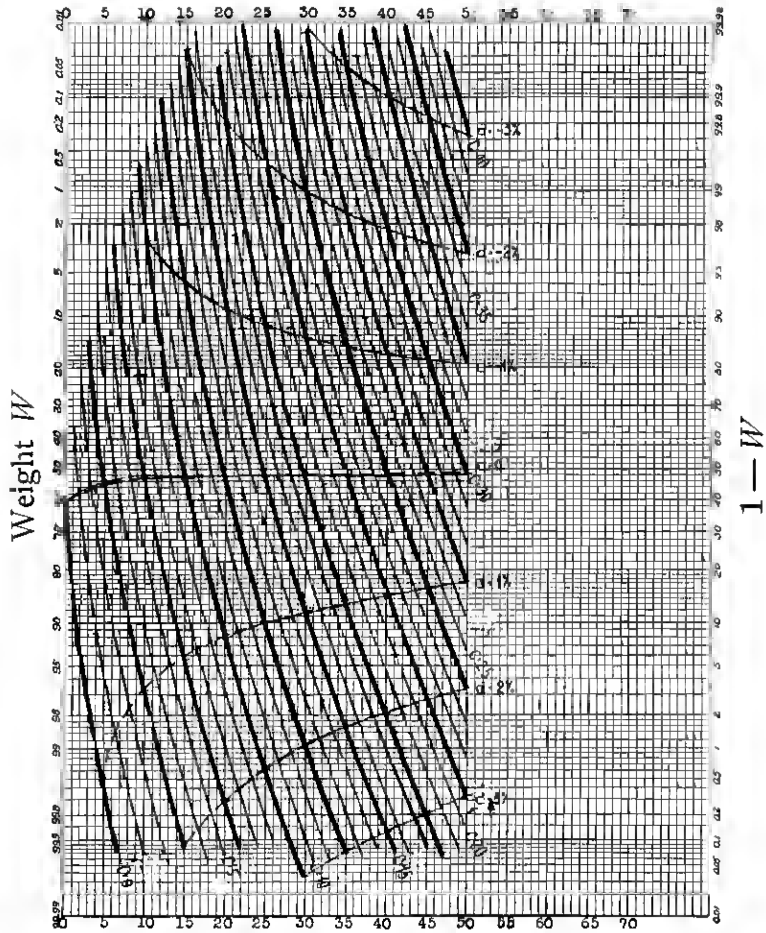
X = Defectives in Universe

$N = 500$, $n = 99$

CHARTS A



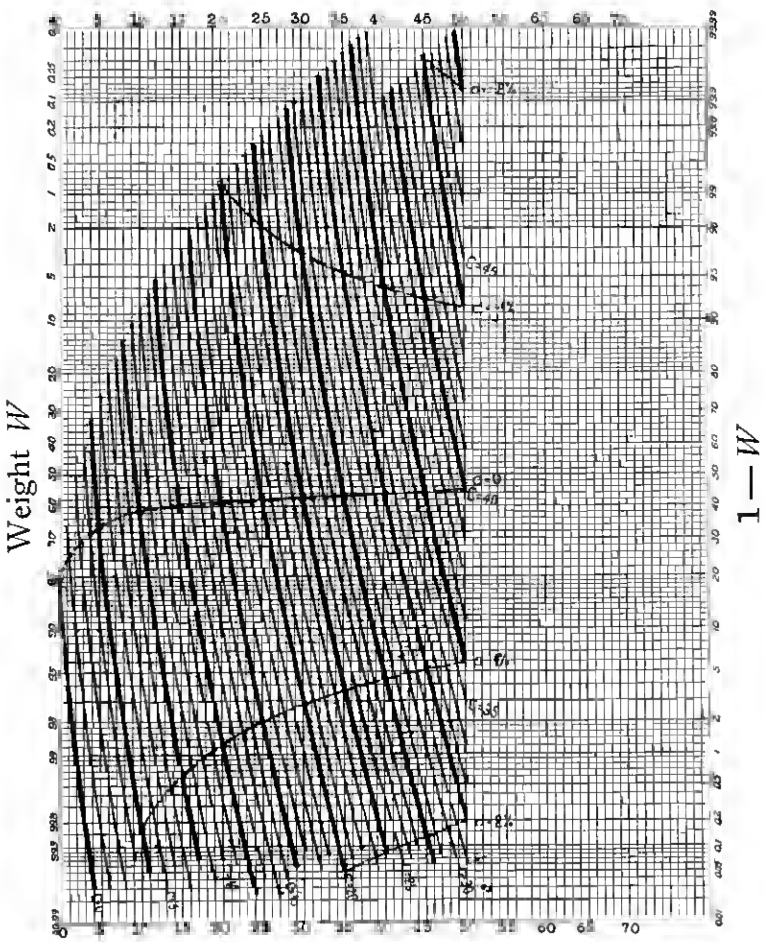
$X =$ Defectives in Universe
$N = 500,\ n = 199$
CHARTS A



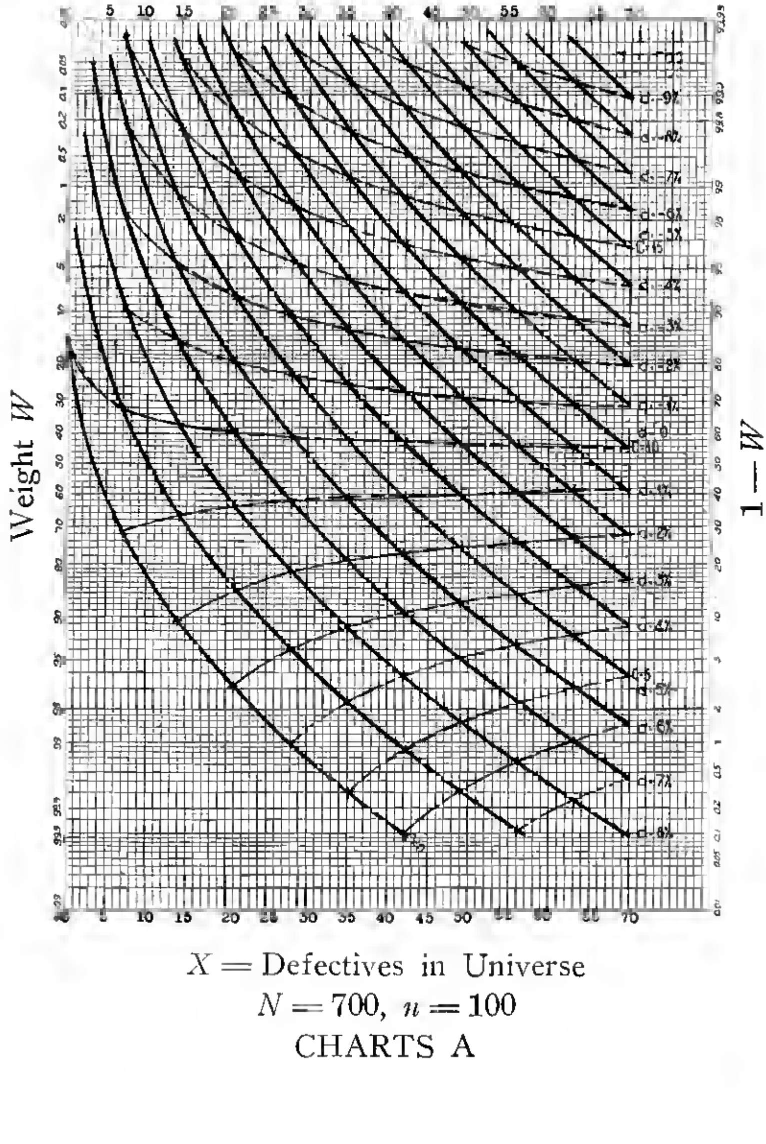
$X =$ Defectives in Universe

$N = 500,\ n = 300$

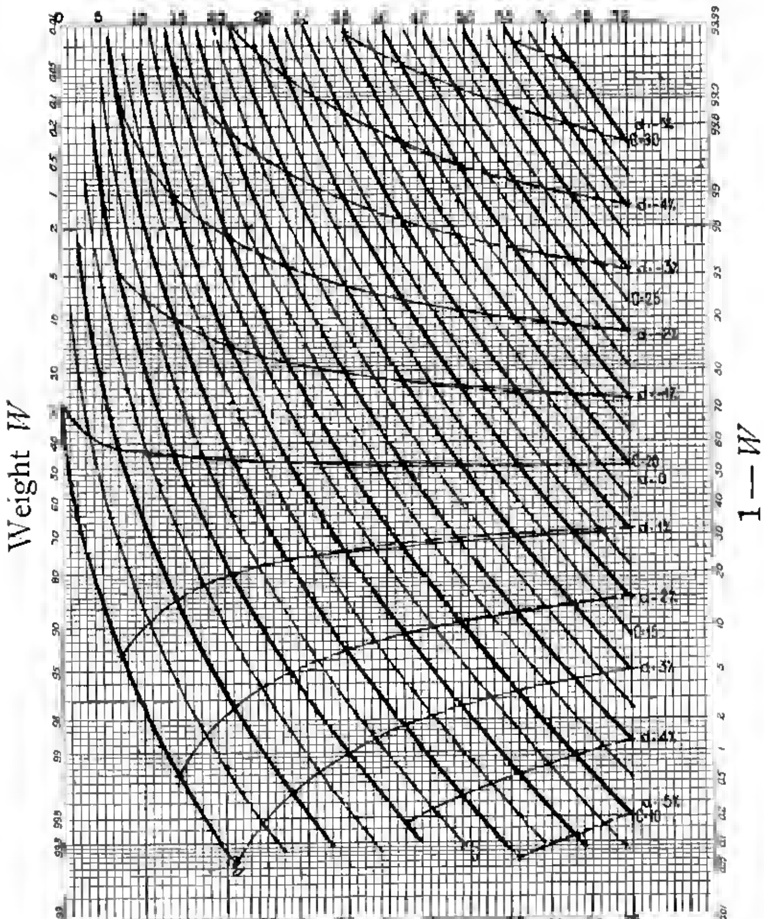
CHARTS A



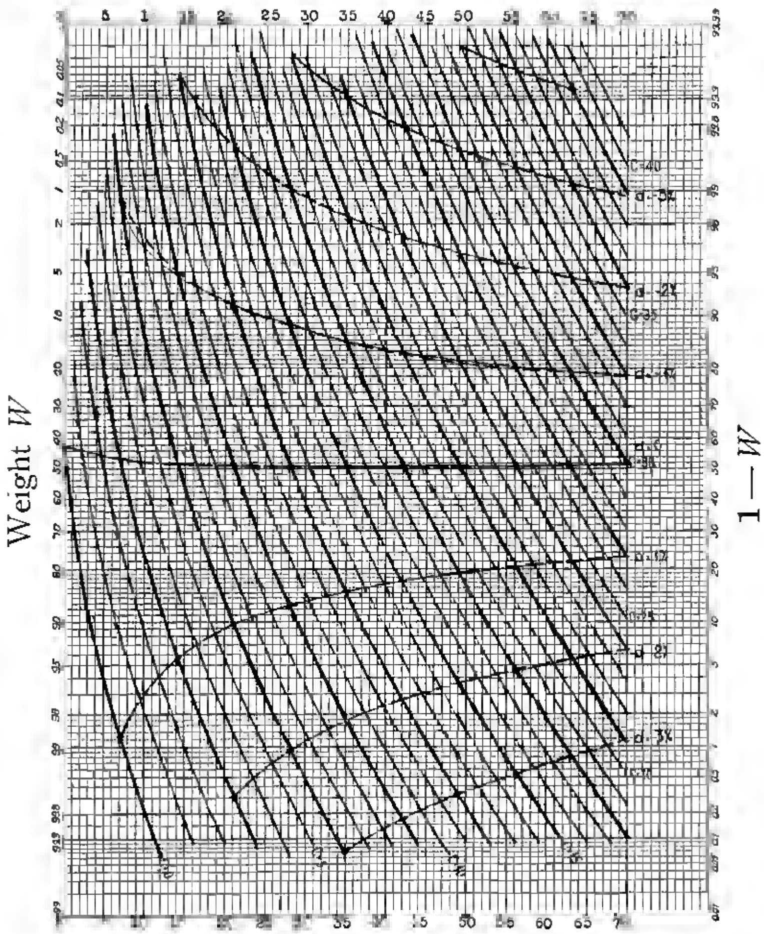
$X =$ Defectives in Universe
$N = 500,\ n = 400$
CHARTS A



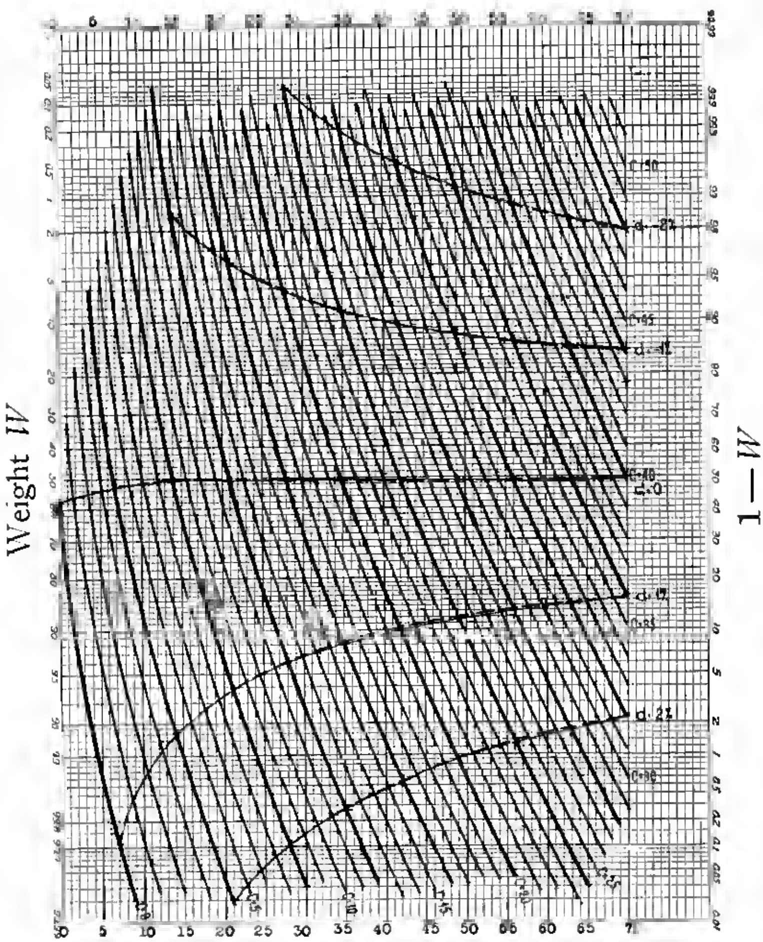
$X=$ Defectives in Universe
$N=700,\ n=100$
CHARTS A



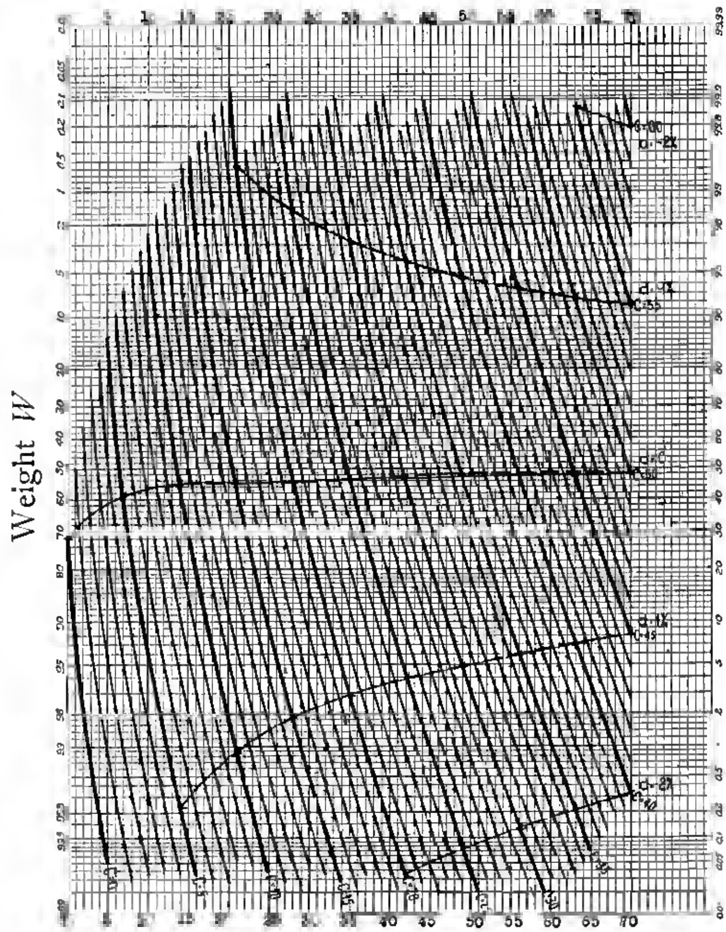
$X=$ Defectives in Universe
$N=700,\ n=200$
CHARTS A



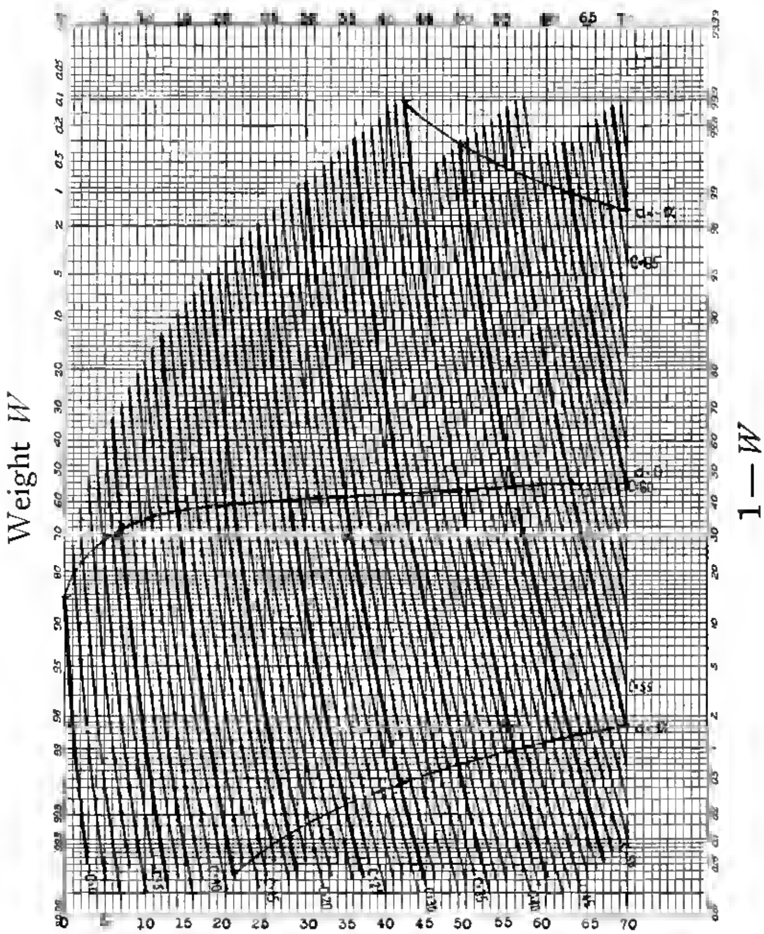
X = Defectives in Universe
$N = 700,\ n = 300$
CHARTS A



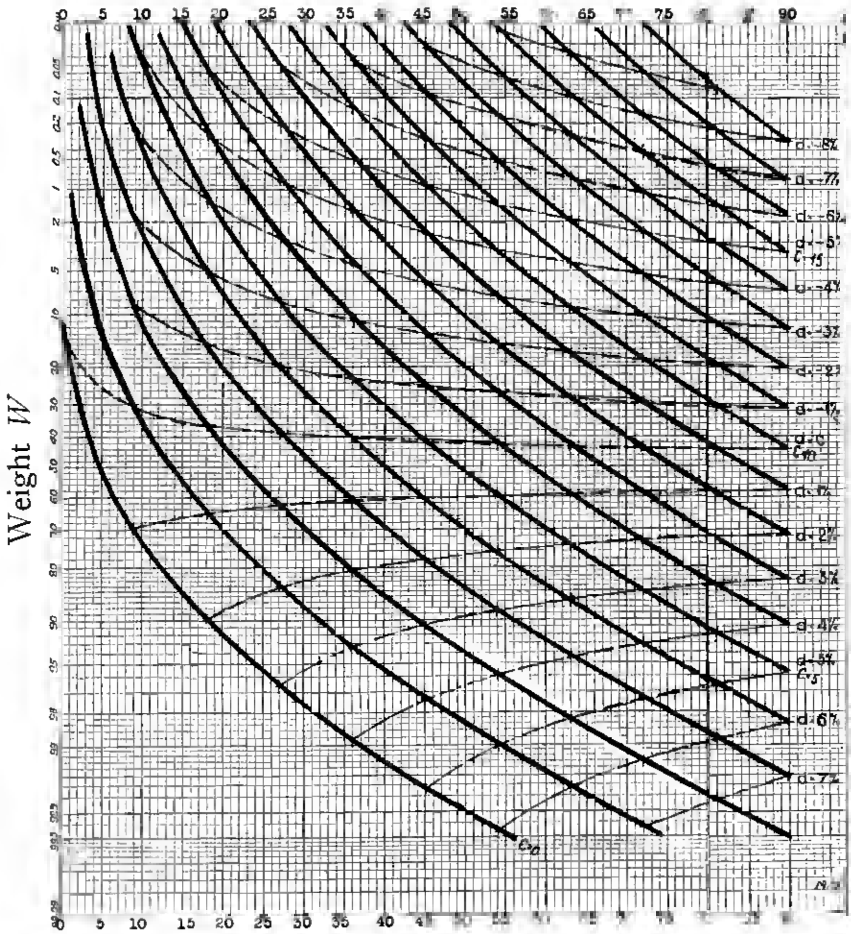
X = Defectives in Universe
$N = 700$, $n = 399$
CHARTS A



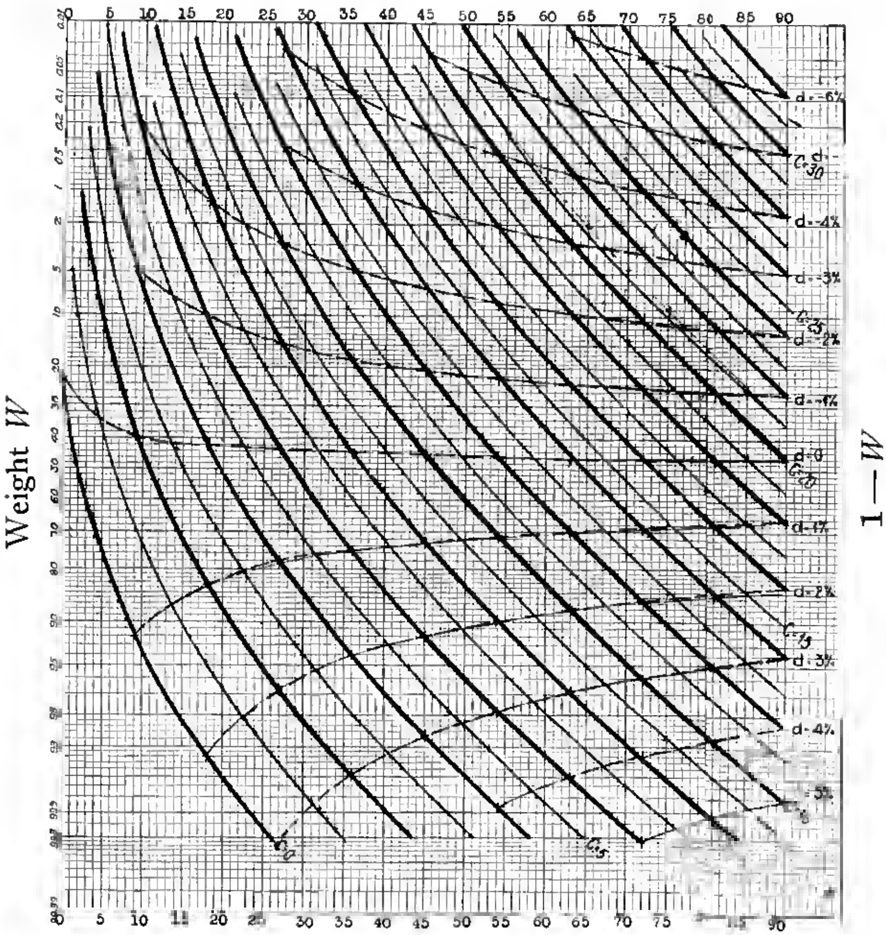
X = Defectives in Universe
$N = 700,\ n = 499$
CHARTS A



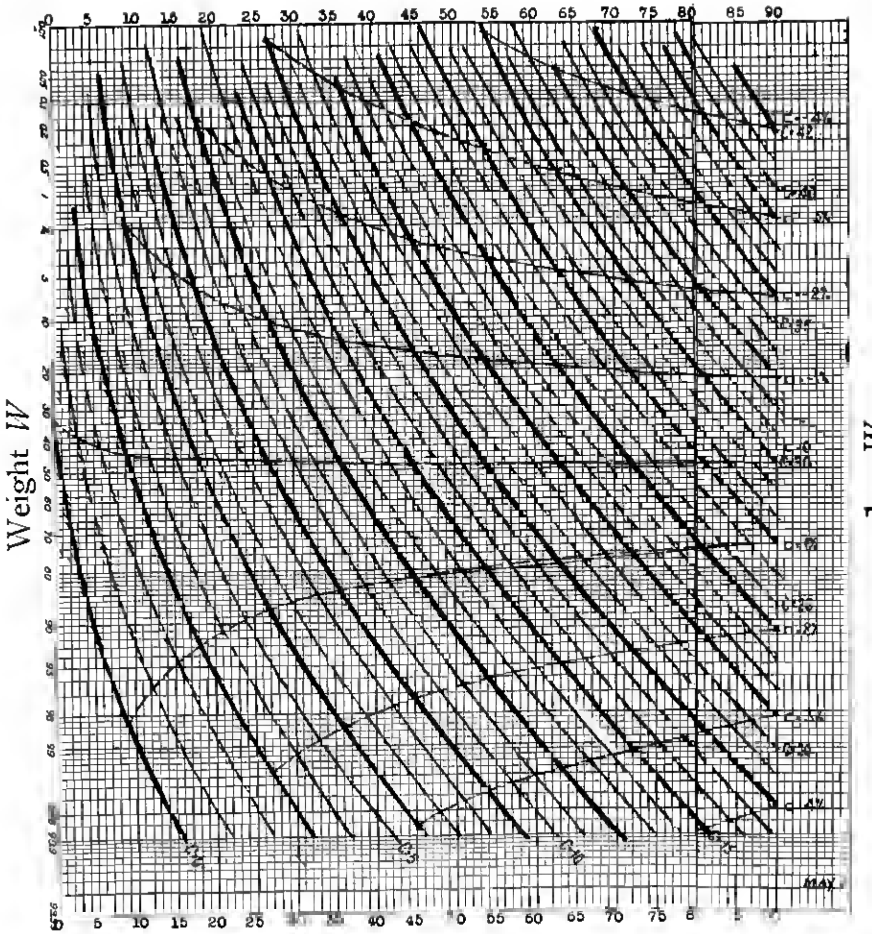
X = Defectives in Universe
$N = 700$, $n = 599$
CHARTS A



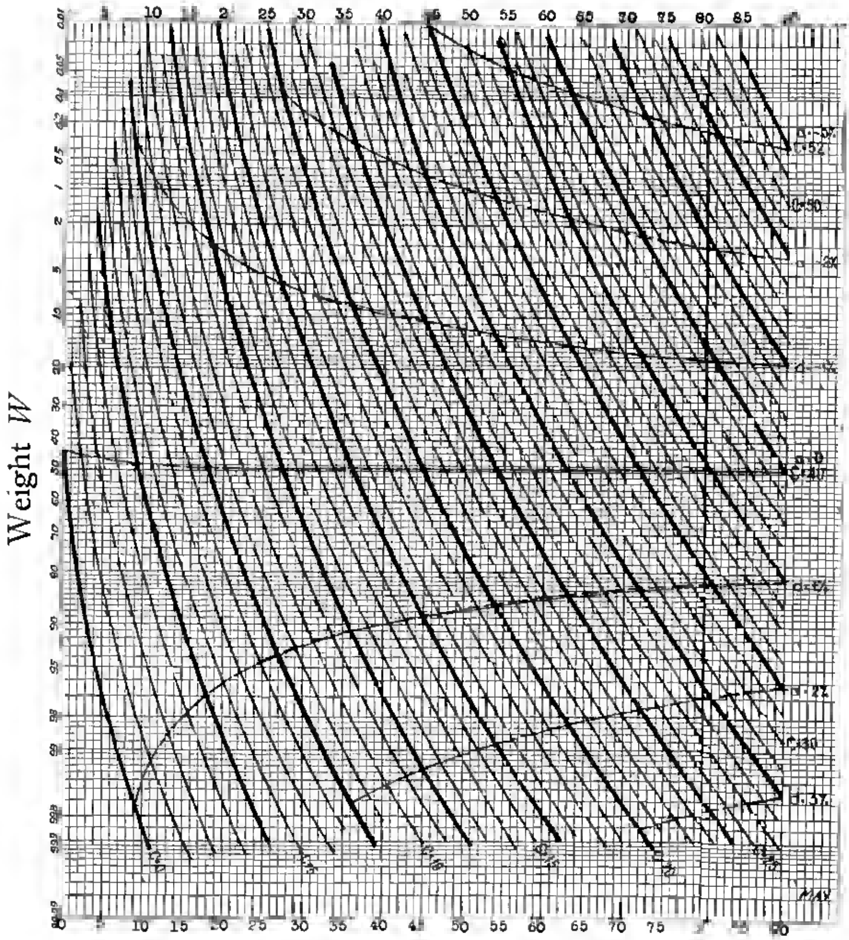
X = Defectives in Universe
$N = 900$, $n = 100$
CHARTS A



$X =$ Defectives in Universe
$N = 900,\ n = 200$
CHARTS A



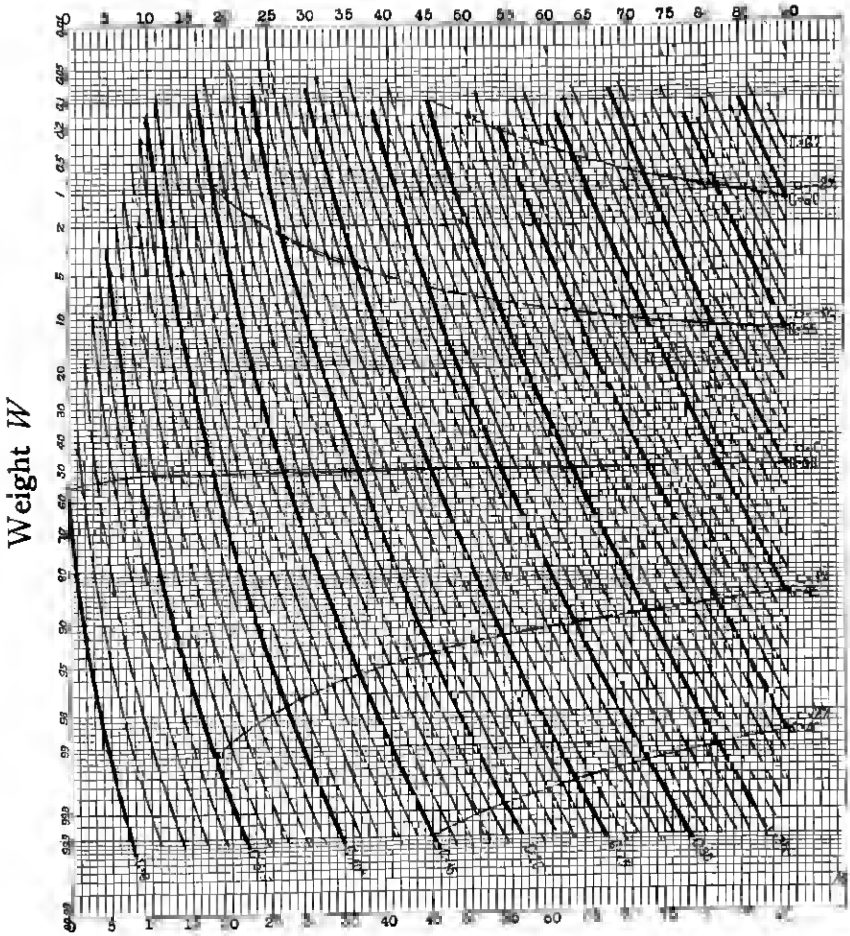
X = Defectives in Universe
$N = 900,\ n = 300$
CHARTS A



X = Defectives in Universe

$N = 900$, $n = 400$

CHARTS A

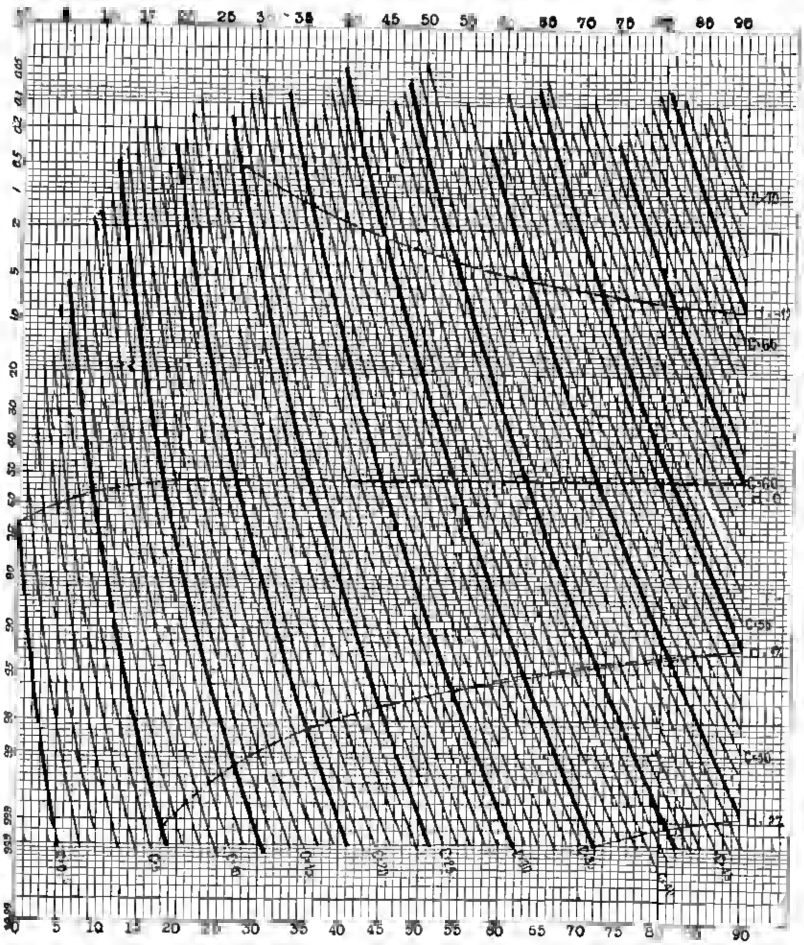


X = Defectives in Universe

$N = 900,\ n = 499$

CHARTS A

Weight W

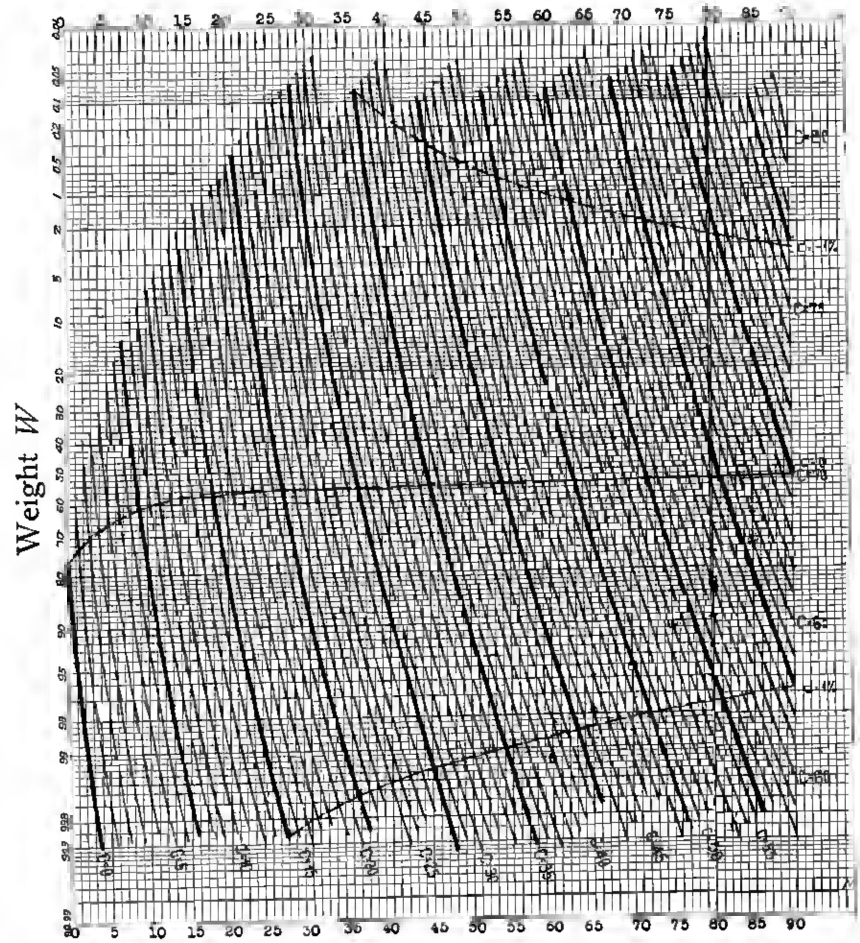


$1 - W$

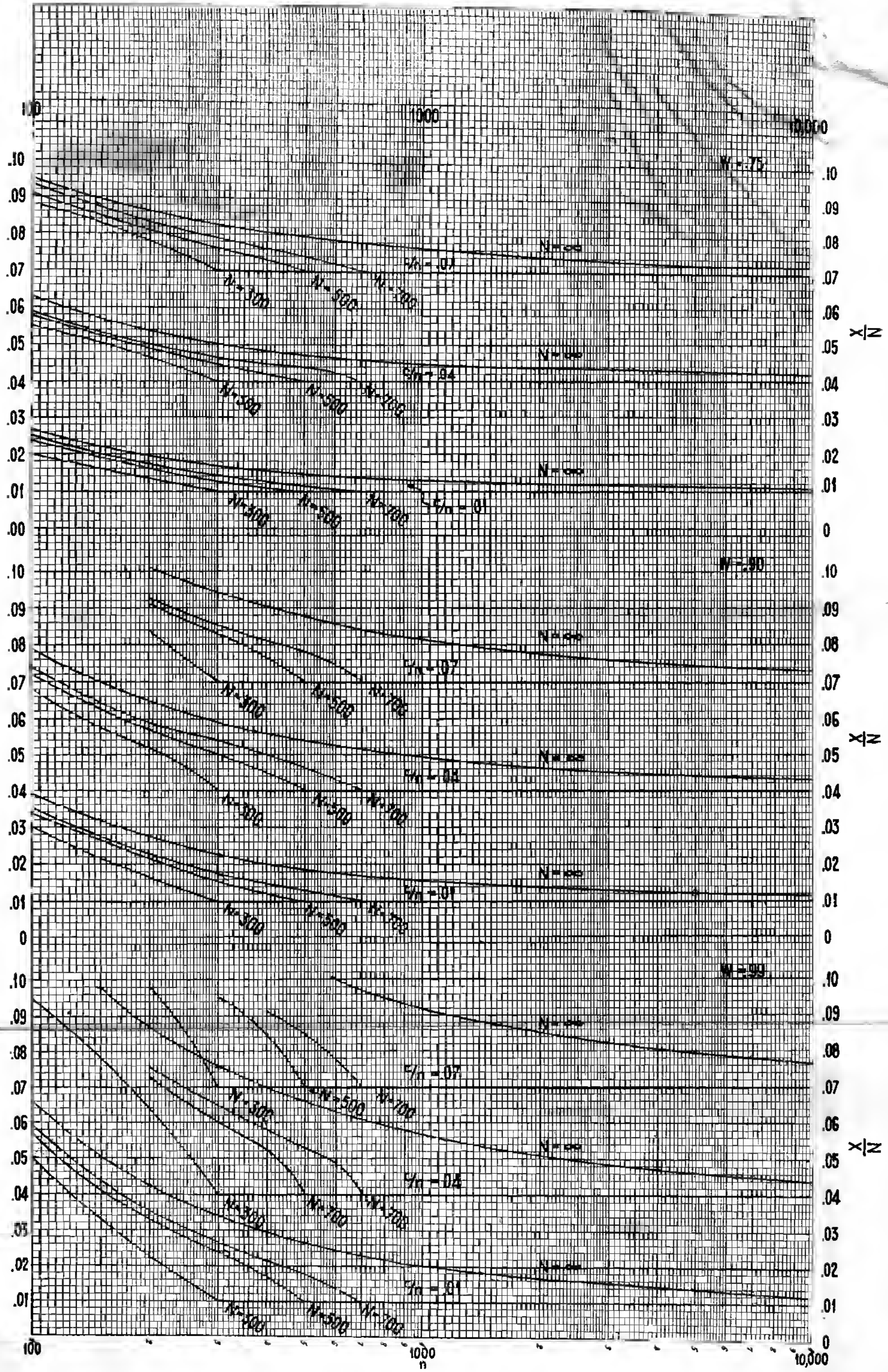
$X =$ Defectives in Universe

$N = 900,\ n = 599$

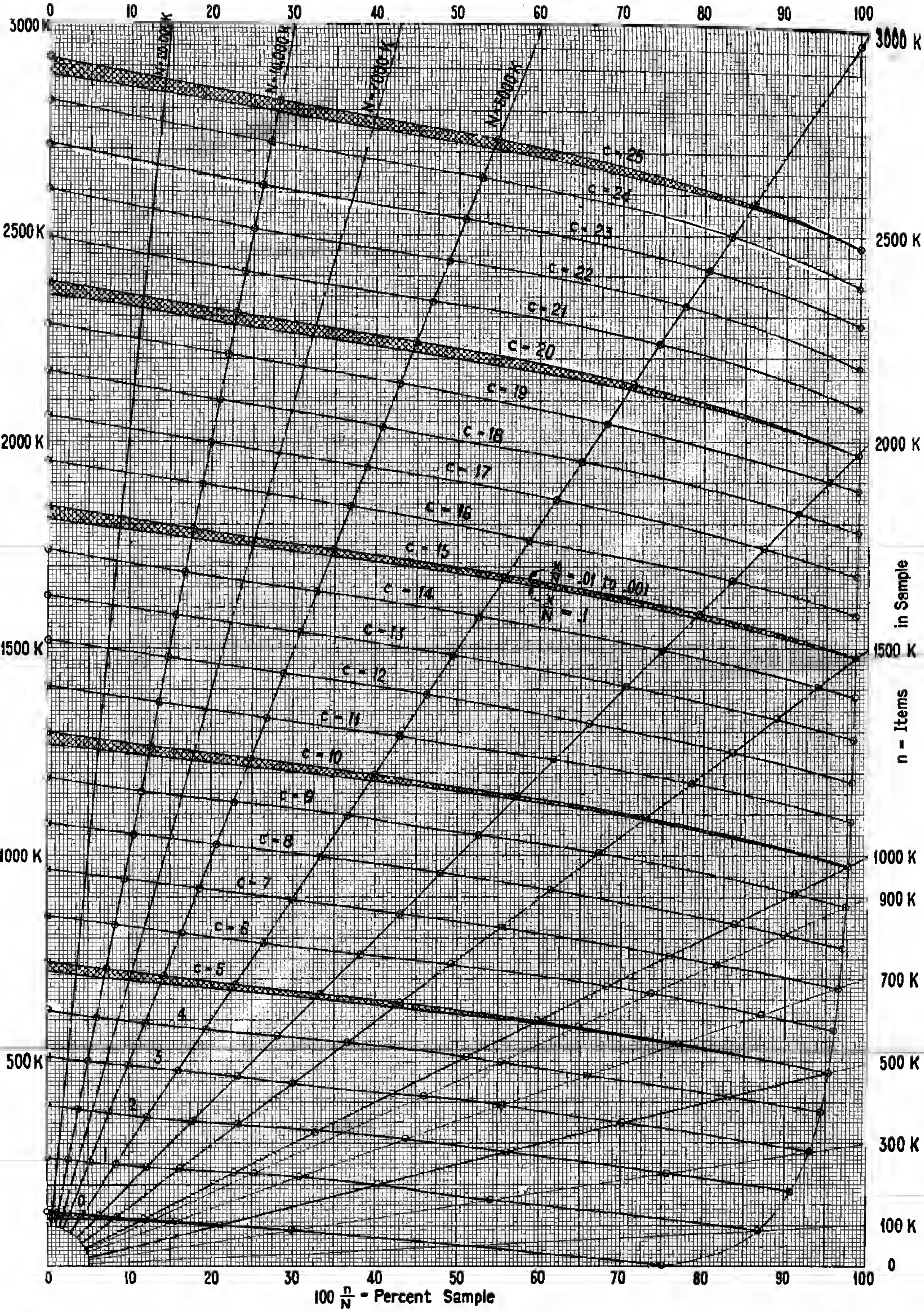
CHARTS A



$X =$ Defectives in Universe
$N = 900$, $n = 699$
CHARTS A

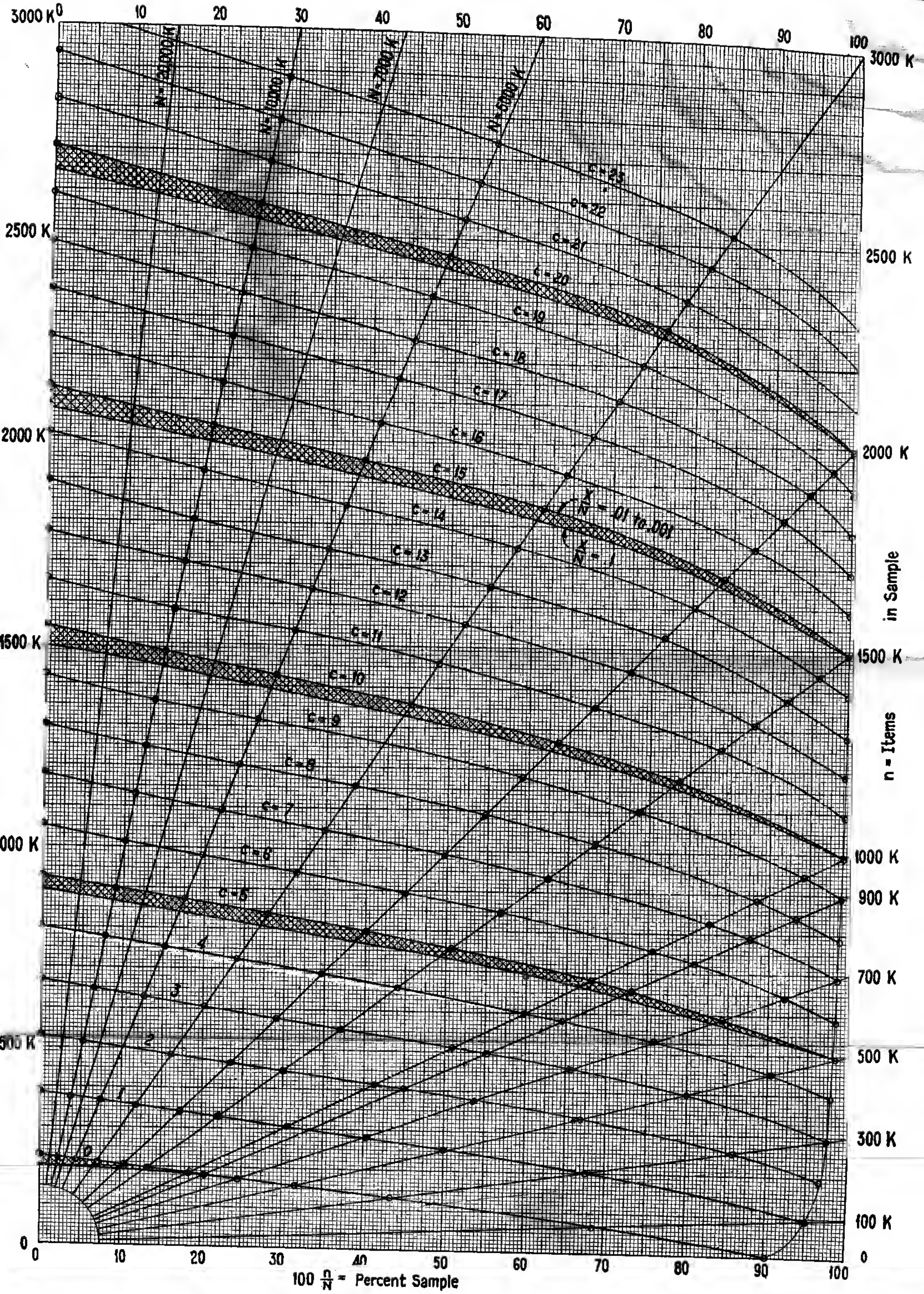


SAMPLING CHART B



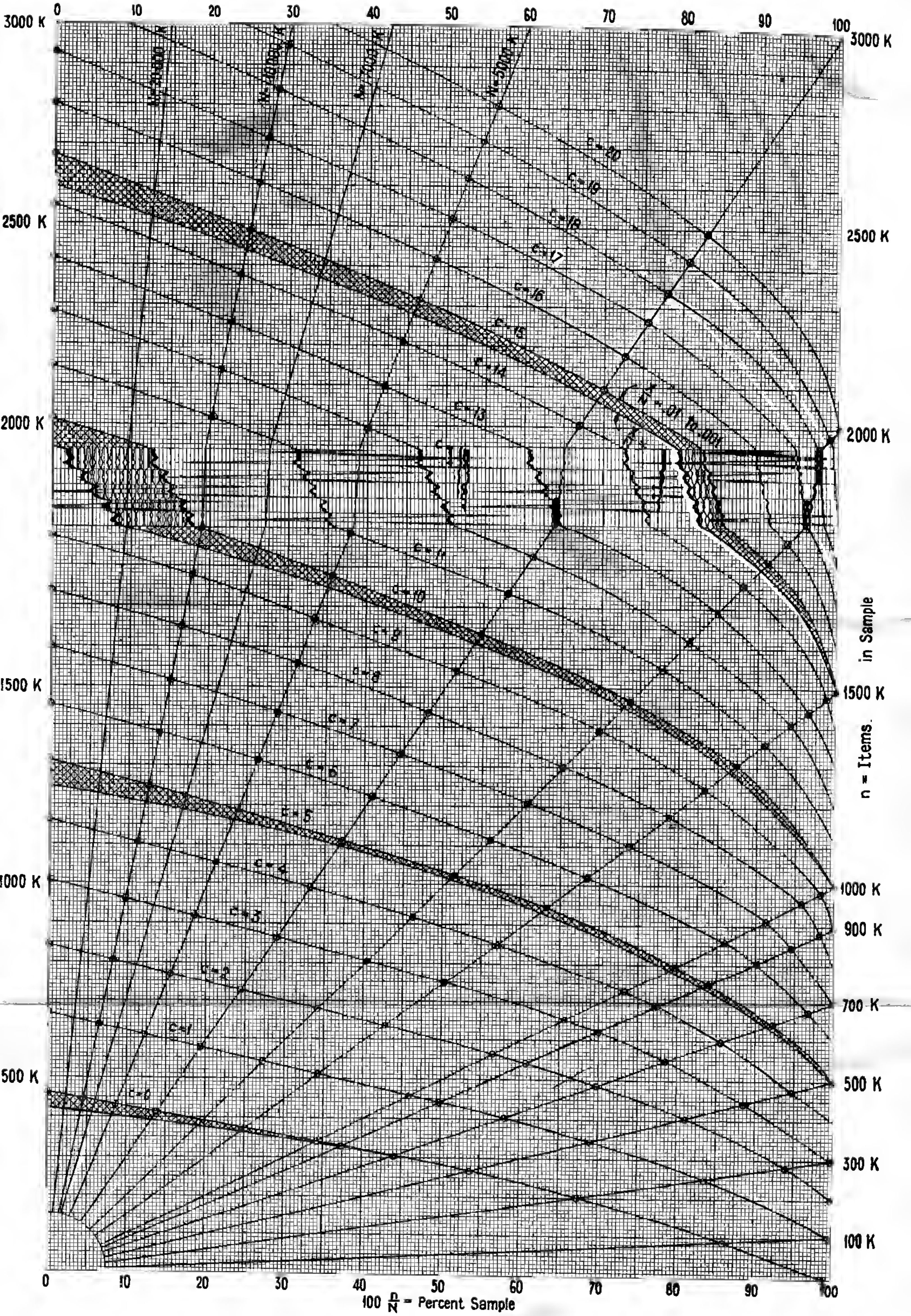
SAMPLING CHARTS C

$$\frac{X}{N} = \frac{.01}{K} \qquad W = .75$$



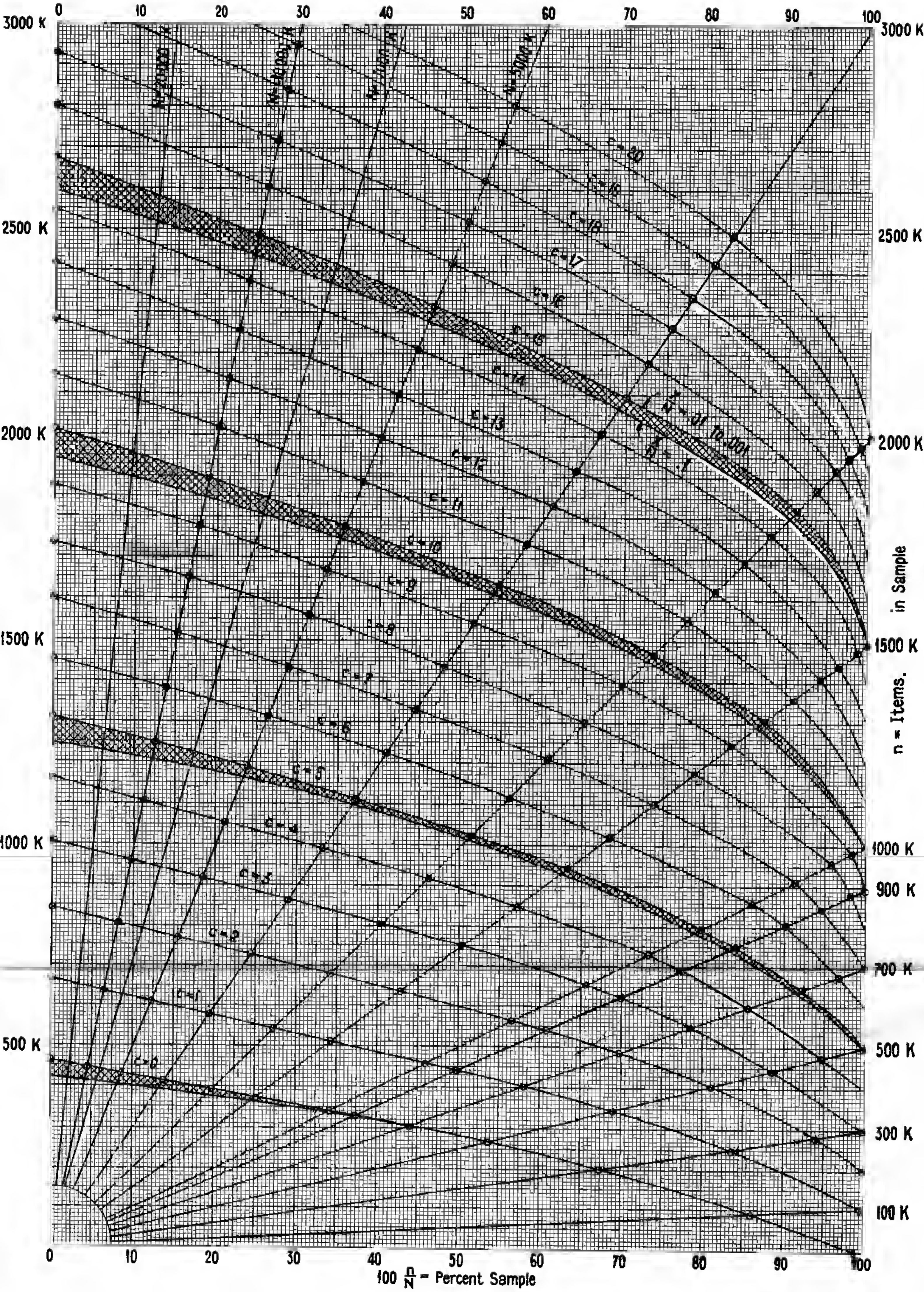
SAMPLING CHARTS C

$$\frac{X}{N} = \frac{.01}{K} \qquad W = .9$$



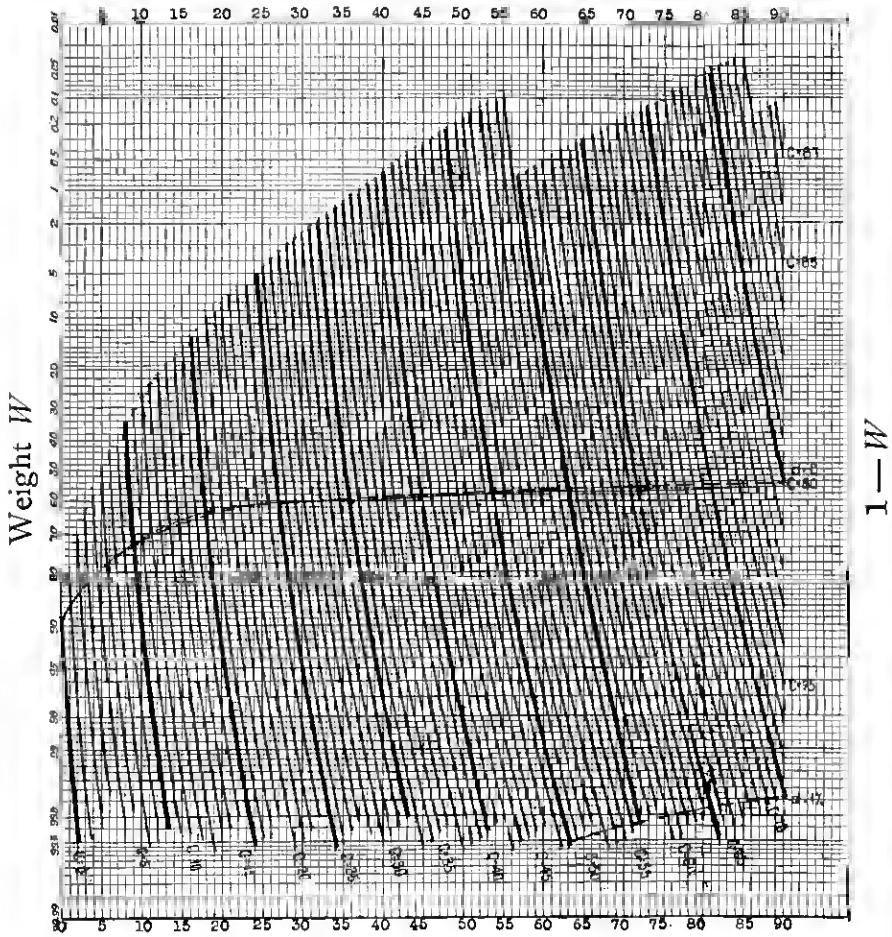
SAMPLING CHARTS C

$$\frac{X}{N} = \frac{.01}{K} \qquad W = .99$$



SAMPLING CHARTS C

$$\frac{X}{N} = \frac{.01}{K} \quad W = .99$$



$X =$ Defectives in Universe

$N = 900,\ n = 799$

CHARTS A